Comment on Two-Fluid Effect and Compressiblity

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1. Introduction

Using the following incompressible two-fluid model,

$$m_i n\left[\frac{\partial \mathbf{u}_i}{\partial t}+\mathbf{u}_i\cdot\nabla\mathbf{u}_i\right]=-\nabla p_i+en\left(\mathbf{E}+\mathbf{u}_i\times\mathbf{B}\right) \qquad (1)$$

$$0=-\nabla p_e-en\left(\mathbf{E}+\mathbf{u}_e\times\mathbf{B}\right) \qquad (2)$$

$$\nabla\cdot\mathbf{u}_i=0, \qquad \nabla\cdot\mathbf{u}_e=0 \qquad (3,4)$$

$$\nabla\times\mathbf{B}=\mu_0 en\left(\mathbf{u}_i-\mathbf{u}_e\right) \qquad (5)$$

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}, \qquad \nabla\cdot\mathbf{B}=0 \qquad (6,7)$$

Yamada et al.[1] have studied the flowing equilibrium. Criteria are found for when the single-fluid model is adequate and when the more general two-fluid model is necessary. The ion flow and plasma beta as well as the size parameter are found to play a major role in the question of whether two-fluid corrections are needed.

The stability of the local mode, which has the infinite azimuthal mode number and is localized on the magnetic flux surface, is studied in the field-reversed configuration (FRC) that is inevitably high beta[2]. It is found in this analysis that the compressibility is not important in spite of the sharp inhomogeneity of magnetic field strength and its curvature almost across the line from the magnetic axis to the x-points on the geometric axis. This tendency is also found in the tilt mode stability analysis in FRCs.[3] From these two studies, the compressibility does not seem to play important role in high beta plasmas such as FRCs. Is it true? If so, why so?

The purpose of this short comment is therefore to clarify t the prediction he two-fluid effect on the eigenmodes and to know the effect of the compressibility comparing derived by the incompressible model with that by the compressible model.

2. Dispersion relation for incompressible two-fluid model

Apply the above incompressible two-fluid model to the non-flowing, uniform equilibrium. Assume that any perturbed quantity $\tilde{f}$ varies as $\tilde{f}\propto\exp(i\mathbf{k}\cdot\mathbf{r}-i\omega t)$, where $\mathbf{k}$ is the wave vector and ω the angular frequency of the mode considered. Then the equations for the perturbation are written as

$$-i\omega m_i n\tilde{\mathbf{u}}_i=-i\mathbf{k}\tilde{p}_i+en(\tilde{\mathbf{E}}+\tilde{\mathbf{u}}_i\times\tilde{\mathbf{B}}) \qquad (8)$$

$$0=-i\mathbf{k}\tilde{p}_e-en(\tilde{\mathbf{E}}+\tilde{\mathbf{u}}_e\times\tilde{\mathbf{B}}) \qquad (9)$$

$$\mathbf{k}\cdot\tilde{\mathbf{u}}_i = 0\,, \qquad \mathbf{k}\cdot\tilde{\mathbf{u}}_e = 0 \tag{10,11}$$

$$i\mathbf{k}\times\tilde{\mathbf{B}} = \mu_0 en(\tilde{\mathbf{u}}_i - \tilde{\mathbf{u}}_e)\,, \quad \mathbf{k}\times\tilde{\mathbf{E}} = \omega\tilde{\mathbf{B}}\,, \quad \mathbf{k}\cdot\tilde{\mathbf{B}} = 0 \tag{12,13,14}$$

Take the curl of eq.(8) and use eq.(13). Then,

$$\tilde{\mathbf{B}} = -\frac{k_z B}{\omega}\tilde{\mathbf{u}}_i - i\frac{m_i}{e}\mathbf{k}\times\tilde{\mathbf{u}}_i \tag{15}$$

Here it is used that the ambient magnetic field is in z-direction. Take the sum of eqs.(8) and (9), and use eq.(12). Then, we have

$$\tilde{p}_i + \tilde{p}_e + \frac{1}{\mu_0}B\tilde{B}_z = 0 \quad \text{and} \quad \omega m_i n\tilde{\mathbf{u}}_i = -\frac{1}{\mu_0}Bk_z\tilde{\mathbf{B}}\,, \tag{16,17}$$

where the relation, $(\mathbf{k}\times\tilde{\mathbf{B}})\times\mathbf{B} = Bk_z\tilde{\mathbf{B}} - B\tilde{B}_z\mathbf{k}$, and eqs. (10, 14) are used. From eqs.(15) and (17), eq.(18) can be derived.

$$\left(\omega^2 - k_z^2 V_A^2\right)\tilde{\mathbf{u}}_i - i\omega V_A \ell_i k_z \mathbf{k}\times\tilde{\mathbf{u}}_i = 0\,, \tag{18}$$

where $\ell_i = c/\omega_{pi}$ is the ion skin depth. Note that the first term is derived from the single-fluid MHD and the second represents the two-fluid effect. Taking the cross product of eq.(18) with the wave vector and using the incompressible condition (10), we have the following eq.(19).

$$i\omega V_A \ell_i k_z k^2 \tilde{\mathbf{u}}_i + \left(\omega^2 - k_z^2 V_A^2\right)\mathbf{k}\times\tilde{\mathbf{u}}_i = 0 \tag{19}$$

From eqs.(18) and (19), the dispersion relation (20) is derived.

$$(Q-1)^2 - (k_z\ell_i)^2\left[1 + (k_\perp/k_z)^2\right]Q = 0\,. \tag{20}$$

Here $Q = (\omega/k_z V_A)^2$ is the normalized frequency squared and $k_\perp^2 = k_x^2 + k_y^2$. As this dispersion relation has two parameters such as $(k_z\ell_i)^2$ and $(k_\perp/k_z)^2$, the eigenmode frequency is characterized by these two parameters. As the dispersion relation (20) is quadratic with respect to Q, two solutions may exist. Figure 1 shows the result. The deviation from the relation, $\omega/k_z V_A = 1$ which represents the so-called sheared Alfven wave, can be significant for fairy small $(k_z\ell_i)^2$ and $(k_\perp/k_z)^2$. This deviation represents the two-fluid effect.

3. Dispersion relation for compressible two-fluid model

To study the effect of the compressibility, let us replace eqs.(10) and (11) by the equations of continuity for each species. Then, we have

$$\tilde{n}_i = \tilde{n}_e \equiv \tilde{n} = -in\mathbf{k}\cdot\boldsymbol{\xi}\,, \tag{21}$$

where $\tilde{\mathbf{u}}_i = -i\omega\boldsymbol{\xi}$. Here we assume for simplicity that the temperature of each species is constant. From sum of eqs.(8) and (9) and use of eq.(21), the equation (22) can be derived.

$$\omega^2\xi - \frac{T_i + T_e}{m_i}\mathbf{k}(\mathbf{k}\cdot\xi) - \frac{iB}{\mu_0 m_i n}\hat{z}\times(\mathbf{k}\times\tilde{\mathbf{B}}) = 0 \qquad (22)$$

where $\hat{z}$ is the unit vector along the ambient magnetic field. Substituting eq.(22) into eq.(8), the perturbed electric field is written as

$$\tilde{\mathbf{E}} = -\frac{m_i}{e}\omega^2\xi + \mathbf{k}\frac{T_i}{e}(\mathbf{k}\cdot\xi) + i\omega\xi\times\mathbf{B} \qquad (23)$$

Substitution of eq.(23) into eq.(13) leads to

$$\tilde{\mathbf{B}} = -\frac{m_i}{e}\omega\mathbf{k}\times\xi + i\mathbf{k}\times(\xi\times\mathbf{B}) \qquad (24)$$

Substituting eq.(24) into eq.(22), we have the eigenvalue equation for the ion displacement ξ,

$$\left(\omega^2 - k_z^2V_A^2\right)\xi + V_A^2k_z\xi_z\mathbf{k} + i\omega\ell_iV_Ak^2\xi_\perp\times\hat{z} - \\ -(\mathbf{k}\cdot\xi)\left[\frac{T_i+T_e}{m_i}\mathbf{k} + V_A^2\mathbf{k}_\perp + i\omega\ell_iV_A\mathbf{k}_\perp\times\hat{z}\right] = 0 \qquad (25)$$

where the subscript $\perp$ denotes the component perpendicular to the ambient magnetic field. The parallel (or z-) component of eq.(25) gives the relation,

$$k_z\xi_z = \frac{(T_i+T_e)k_z^2}{\omega^2 m_i}(\mathbf{k}\cdot\xi) \qquad (26)$$

Taking the inner and cross products of eq.(25) with the wave vector and using eq.(26), we have the following coupled equations

$$i\omega\ell_iV_Ak^2\mathbf{k}\cdot(\xi_\perp\times\hat{z}) + \left[\omega^2 - k^2\left(V_A^2 + C_s^2\right) + k^2C_s^2\frac{k_z^2V_A^2}{\omega^2}\right](\mathbf{k}\cdot\xi) = 0, \qquad (27)$$

$$\left(\omega^2 - k_z^2V_A^2\right)\mathbf{k}\cdot(\xi_\perp\times\hat{z}) - i\omega k_z^2\ell_iV_A\left(1 - \frac{k^2C_s^2}{\omega^2}\right)(\mathbf{k}\cdot\xi) = 0, \qquad (28)$$

where $C_s = \left(T_i + T_e/m_i\right)^{1/2}$. From these equations the dispersion relation can be written as

$$\left\{\beta - \left[1 + \left(k_\perp/k_z\right)^2\right]^{-1}Q\right\}\left\{(Q-1)^2 - \left(k_z\ell_i\right)^2\left[1 + \left(k_\perp/k_z\right)^2\right]Q\right\} + \\ + \frac{\left(k_\perp/k_z\right)^2}{1 + \left(k_\perp/k_z\right)^2}Q\,(Q-1) = 0 \qquad (29)$$

Compared with the dispersion relation (20), this dispersion relation has the additional parameter $\beta = n(T_i + T_e)\big/(B^2/\mu_0)$, which denotes the plasma beta. Note that the second curly bracket is the same as the left hand side of eq.(20). As the dispersion relation (29) is cubic with respect to Q, three solutions may exist. Note also that the dispersion relation (29) can be derived from the coupled equations for the electrostatic potential and the parallel component of the vector potential[4] although its derivation is more complex.

When $(k_\perp/k_z)^2 = 0$, i.e. the wave propagate along the ambient magnetic field, the second term of eq.(29) vanishes, resulting in the decoupling of the incompressible shear Alfven mode with two-fluid correction and the following acoustic wave

$$\omega = k_z C_s \qquad (30)$$

Note that this mode can not described by the incompressible dispersion relation (20).

Tables 1 and 2 show the solutions of the dispersion relation (29) for various values of $(k_z \ell_i)^2$, $(k_\perp / k_z)^2$ and β. As shown on the foot note, the frequencies with superscripts a) and b) are derived by the incompressible dispersion relation (20) and the frequencies with the superscript c) are derived by the vanishing condition of the first curly bracket of the compressible dispersion relation (29). At first look at the extremely high beta case such as β=100 in Table 1. Each of the three solutions of eq.(29) is in high accuracy equal to the corresponding frequency with the superscript a) or b) or c). This means that the second term of eq.(29) is not important for the extremely high beta case and as a result, the sheared Alfven wave predicted by the incompressible dispersion relation (20) works well. This tendency is also observed in Table 2. Actually, for high beta plasmas with $\beta \geq 3$, this tendency can be observed. For low beta plasmas such as $\beta \leq 0.01$, however, the second term of eq.(29) is important and as a result, the incompressible dispersion relation (20) can not be applicable.

4. Summary and Conclusion

Applying the two-fluid model to an uniform plasma, we have studied the two-fluid effect and the compressibility. As shown in Fig.1, the frequency of the sheared Alfven mode, $\omega = k_z V_A$, is significantly modified by the two-fluid effect for not so large values of $(k_z \ell_i)^2$ and $(k_\perp / k_z)^2$. Comparison between the incompressible and compressible models reveals that for high beta plasmas with $\beta \geq 3$, the incompressible model is useful to study the sheared Alfven-like mode, but for low beta plasmas with $\beta \leq 0.01$, the incompressible model cannot be applicable.

In FRCs, the average beta defined by $\left\langle p \big/ (p + B^2 / \mu_0) \right\rangle$ can be large as 0.75-0.9. This means $\beta \geq 3$. Hence, the above results are consistent with the results found in Refs.2 and 3 as described in Introduction.

References

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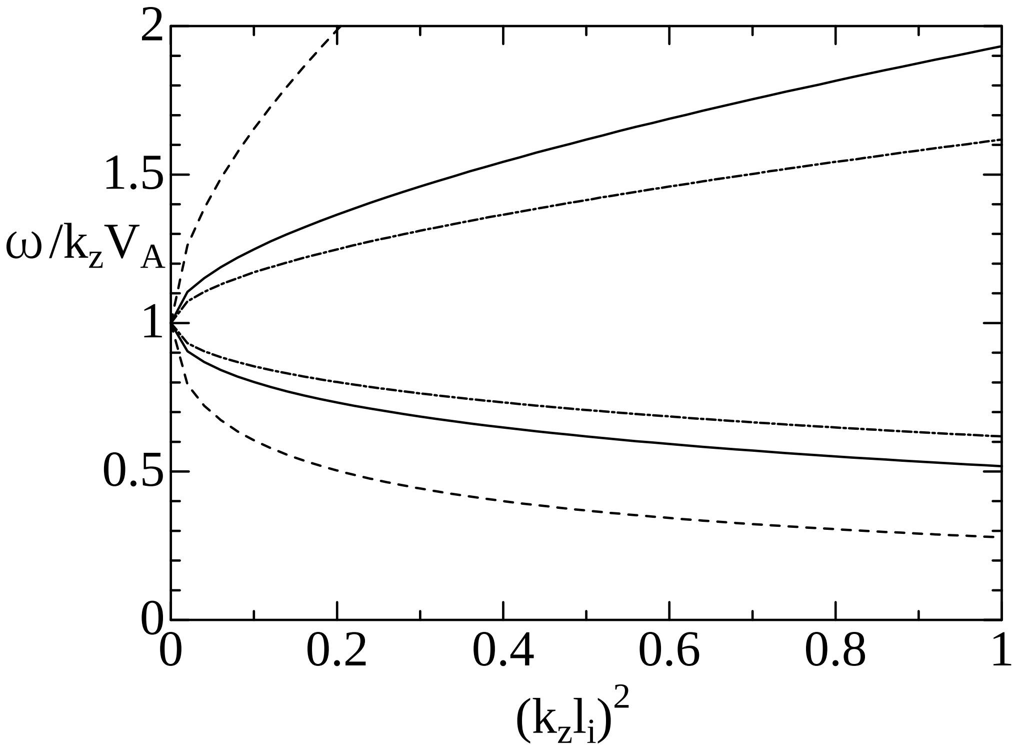

Fig.1 ω/k_zV_A versus $(k_z\ell_i)^2$ for $(k_\perp/k_z)^2$=0.0 (dot-dashed line), 1.0 (solid line) and 10.0 (broken line)

Table 1

$(k_z\ell_i)^2$	$(k_\perp/k_z)^2$	β	ω/k_zV_A			
0.2	1.0	0.01	0.0996	0.871	1.628	(0.141 [c])
0.2	1.0	0.10	0.305	0.891	1.647	(0.447 [c])
0.2	1.0	1.0	0.654	1.122	1.927	(1.414 [c])
0.2	1.0	3.0	0.709	1.282	2.693*	(2.449 [c])
0.2	1.0	10.0	0.726	1.342	4.590*	(4.472 [c])
0.2	1.0	100.0	0.732	1.363	14.18*	(14.14[c])
0.2	1.0		0.733[a]	1.365 [b]		

[a),b)] These frequencies are derived by the incompressible dispersion relation (20).

[c)] These frequencies are derived by the vanishing condition of the first curly bracket of the compressible dispersion relation (29).

*) These frequencies are larger than ω_{ci} because of $\omega/\omega_{ci} = (k_z\ell_i)(\omega/k_zV_A)$.

Table 2

$(k_z\ell_i)^2$	$(k_\perp/k_z)^2$	β	ω/k_zV_A			
0.0001	1.0	0.01	0.0316	1.000	1.415	(0.141 [c])
0.0001	1.0	0.10	0.308	1.000	1.451	(0.447 [c])
0.0001	1.0	1.0	0.765	1.000	1.848	(1.414 [c])
0.0001	1.0	3.0	0.915	1.000	2.676	(2.449 [c])
0.0001	1.0	10.0	0.973	1.002	4.588	(4.472 [c])
0.0001	1.0	100.0	0.992	1.006	14.18	(14.14[c])
0.0001	1.0		0.993[a]	1.007 [b]		

[a),b)] These frequencies are derived by the incompressible dispersion relation (20).

[c)] These frequencies are derived by the vanishing condition of the first curly bracket of the compressible dispersion relation (29).