Effect of angular momentum on FRC minimum energy states

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Abstract

All stable equilibria are stationary-energy (SE) states of some form. Thus finding the SE states considerably simplifies the search for stable plasma equilibria. This is particularly so in the case of flowing equilibria for which finding any kind of equilibrium is exceedingly complicated. A previous analysis found SE of a two fluid plasma by minimizing the ordered energy W_{MF} (flow + magnetic field energy) subject to constraints on the two integral constants of motion for a two-fluid, the ion helicity K_i and the electron helicity K_e .

The foregoing analysis has been extended to include the effect of another constant of motion, the angular momentum, L_z. In the case of an axisymmetric system with suitable boundary conditions (constant magnetic flux boundary, no flow through the boundary, free-slip condition) the angular momentum is an integral constant of motion. Incorporating this additional constraint leads to a more complicated system of equations. Instead of a two-point spectrum of SE states, all *axisymmetric* mode elements are represented except in exceptional cases, although the spectrum tends to concentrate in certain ranges. An important consequence of angular momentum conservation is that the solution is axisymmetric, i.e., tilted equilibrium states are disallowed.

Inspection of the form of the spectrum reveals potential pathological behavior, which must be carefully dealt with in numerical calculations. A method for removing the pathologies is presented and applied. Investigation of the equations indicates that there are two regimes for the SE states, a "no-root" region in which all modes contribute to the conserved quantities, and a "root" region in which the spectrum may collapse to a one- or two-point spectrum. Calculations of the functional form W_{MF} (K_i,K_e,L_z) are presented for both cases.

I. Introduction

In previous work [1] the SE states for a two-fluid flowing plasma were obtained, using the variational technique to minimize the ordered energy, W_{MF}, subject to the constraint that ion and electron helicities, K_i and K_e , are conserved. By expanding the flow and field vectors in a complete basis set of divergence-free vectors, known as Beltrami modes, the problem was reduced to a system of algebraic equations. Solution of these equations led to the prediction of SE states that are a two-point spectrum of the basis set (i.e. *double-mode condensates*) and provided the explicit functional form for the stationary energy, W_{MF} $(K_i,K_e,\Lambda_1,\Lambda_2)$. These states included so-called tilted (non-axisymmetric) modes. The values of $K_i,K_e,\Lambda_1,\Lambda_2$ which yield the global minimum for W_{MF} define the stable state.

The objective of this paper is to determine what effect angular momentum conservation has upon the SE states. In particular, we address the question: does the constraint L_z = constant predict a different state to have the lowest W_{MF} than that obtained without angular momentum conservation? If the answer is "yes", then angular momentum conservation has significant implications for plasma stability.

The outline of this paper is as follows. Section II briefly describes the variational procedure and the resulting algebraic equations for the stationary states. Section III explains and resolves the mathematical pathologies of the algebraic variational equations. Section IV, along with Appendix A, presents the results for angular momentum conservation and compares it with the non-conserved case.

II. Variational Procedure for FRC

Application of the variational method to the FRC is done with the following steps:

1. determine the ideal invariants
2. minimize the ordered energy, subject to the constraint that the invariants are conserved, obtaining the Euler-Lagrange equations
3. expand the flow velocity and vector potential in Beltrami modes, converting the vector differential Euler-Lagrange equations into algebraic equations
4. solve for ordered energy as a function of the ideal invariants by eliminating the Lagrange parameters;
5. find the state with the lowest WMF, i.e., the stable state.

Based on the approach of ref. 1, the ordered energy, W_{MF}, and ideal invariants, K_i, K_e, and L_z are given by the volume integrals:

ordered energy: $$W_{MF} = \int d\tau \left(m_i n u_i^2/2 + B^2/8\pi\right) \tag{1a}$$

ion helicity: $$K_i = \left(c^2/8\pi e^2\right)\int d\tau\, \mathbf{P}_i \cdot \nabla\times\mathbf{P}_i \tag{1b}$$

electron helicity: $$K_e = \left(1/8\pi\right)\int d\tau\, \mathbf{A}\cdot\nabla\times\mathbf{A} \tag{1c}$$

angular momentum: $$L_z = \int d\tau\, m_i n\, r u_{i\theta} \tag{1d}$$

Note that the ion canonical momentum is $\mathbf{P}_i = m_i\mathbf{u}_i + e\mathbf{A}/c$ and, for massless electrons, $\mathbf{P}_e = -e\mathbf{A}/c$. Also note that the vector potential terms appearing in the angular momentum ($\boldsymbol{r}\times(\mathbf{P}_i + \mathbf{P}_e)$) cancel out.

Now, assume uniform density and extremize W_{MF} subject to constraints on K_i, K_e, L_z; this yields the Euler equations in matrix-differential-operator format:

$$\overleftrightarrow{D}\begin{Bmatrix} c\mathbf{B}/4\pi e n \\ \mathbf{u}_i \end{Bmatrix} = -\Omega r\hat{\theta}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \tag{2a}$$

$$\vec{D}(\nabla,\lambda_e,\lambda_i) \equiv \begin{bmatrix} \nabla\times-\lambda_e & -1 \\ \lambda_i & \ell_i^2\lambda_i\nabla\times-1 \end{bmatrix} \tag{2b}$$

Note the appearance of the ion skin depth, $\ell_i = (m_i c^2/4\pi e^2 n)^{1/2}$, which is the natural length scale always appearing in two-fluid analyses. The Lagrange multipliers here are λ_e for K_e, λ_i for K_i, and Ω for L_z.

Now expand the fields and flows in a complete, orthogonal set of basis vectors, the eigenfunctions of the curl (Beltrami functions): $\nabla\times\mathbf{Y}_k = \Lambda_k\mathbf{Y}_k$ with boundary condition $\mathbf{Y}_k\cdot\hat{\mathbf{n}} = 0$. For our 3D cylindrical domain, $k = (l,m,n)$ is a 3D index. The expansions of the field and flow in the Beltrami functions are

$$\mathbf{A} = \sum A_j\mathbf{Y}_j\,, \qquad \mathbf{u}_i = \sum u_j\mathbf{Y}_j\,; \tag{3}$$

where A_j, u_j are the expansion coefficients. Substituting these into (2a, b) and taking the *inner product* of each equation with the eigenvector $\mathbf{Y}_k$, i.e. $\int d\tau\mathbf{Y}_k\cdot(\ldots)$ gives the algebraic equations:

$$\vec{D}_k\begin{Bmatrix}(c/4\pi en)\Lambda_k A_k \\ u_k\end{Bmatrix} = -\Omega a\Theta_k\begin{Bmatrix}1\\1\end{Bmatrix}; \tag{4a}$$

$$\vec{D}_k(\lambda_i,\lambda_e) = \begin{bmatrix}\Lambda_k-\lambda_e & -1 \\ \lambda_i & \lambda_i\Lambda_k\ell_i^2-1\end{bmatrix} \tag{4b}$$

where we have defined the geometric factor (a = cylinder radius and V = volume):

$$\Theta_k \equiv (1/aV)\int d\tau\, r\hat{\boldsymbol{\theta}}\cdot\mathbf{Y}_k = \frac{1}{p_l}\frac{\sin(n\pi/2)}{(n\pi/2)}$$

where p_l is the l^{th} root of J_1. Solving (4a, b) for the modal coefficients:

$$\frac{eA_k}{m_i c} = -\frac{\lambda_i}{D_k}\Omega a\Theta_k\,; \qquad u_k = \frac{\lambda_e+\lambda_i-\Lambda_k}{D_k}\Omega a\Theta_k \tag{5a}$$

where
$$D_k \equiv \left\|\vec{D}_k\right\| = (\Lambda_k-\lambda_e)(\Lambda_k\lambda_i\ell_i^2-1)+\lambda_i \tag{5b}$$

and
$$\Lambda_k = (1/a)\sqrt{p_l^2+(n\pi a/L)^2}\,. \tag{5c}$$

This spectrum contains *only axisymmetric elements* since Θ_k is zero except for axisymmetric eigenvectors. Hence, tilted modes are eliminated for conserved angular momentum and $k = (l,0,n)$.

Substituting the coefficients (5a) into (3) and then into the integrals (1a) through (1d) gives:

$$K_e = \frac{\tilde{\Omega}^2}{8\pi}F_e(\lambda_e,\lambda_i); \qquad F_e \equiv \frac{1}{V}\sum_k\frac{\lambda_i^2}{D_k^2}\Lambda_k a^2\Theta_k^2 \tag{6a}$$

$$K_i = \frac{\tilde{\Omega}^2}{8\pi}F_i(\lambda_e,\lambda_e); \qquad F_i \equiv \frac{1}{V}\sum_k\frac{(\Lambda_k-\lambda_e)^2}{D_k^2}\Lambda_k a^2\Theta_k^2 \tag{6b}$$

$$L_z = \frac{en}{c}\tilde{\Omega}F_z(\lambda_e,\lambda_i); \qquad F_z \equiv \frac{1}{V}\sum_k\frac{\lambda_e+\lambda_i-\Lambda_k}{D_k}a^2\Theta_k^2 \tag{6c}$$

$$W_{MF} = \frac{\tilde{\Omega}^2}{8\pi}F_W(\lambda_e,\lambda_e); \; F_W \equiv \frac{1}{V}\sum_k\frac{(\lambda_e+\lambda_i-\Lambda_k)^2+\Lambda_k^2\lambda_i^2\ell_i^2}{\ell_i^2 D_k^2}a^2\Theta_k^2 \tag{6d}$$

with $\tilde{\Omega} \equiv m_i c\Omega / e$. The next step is to remove the dependence on the angular momentum parameter $\tilde{\Omega}$ by normalizing in the helicity magnitude and defining the remaining dimensionless terms as follows:

$$\tan(\theta) = \frac{K_i}{K_e} \tag{7a}$$

$$R_z = \frac{\gamma}{2m_i nVa}\frac{L_z^2}{|K|} = \frac{\gamma}{2M_{tot}a}\frac{L_z^2}{|K|} \tag{7b}$$

$$R_W = \frac{l_i W_{MF}}{|K|} \tag{7c}$$

where $\gamma = l_i / a$ is the system size parameter. Finally, we scale all lengths in ion skin depth and define the transformed Lagrange parameters, ζ and η, to get the dimensionless forms of the sums (6a) through (6d):

$$\eta \equiv \frac{1}{2}\left(\frac{1}{l_i\lambda_i} - l_i\lambda_e\right); \qquad \zeta \equiv \frac{1}{2}\left(\frac{1}{l_i\lambda_i} + l_i\lambda_e\right)$$

$$F_e(\eta,\zeta) \equiv \sum_k \frac{x_k\Theta_k^2}{\Delta_k^2}; \tag{8a}$$

$$F_i(\eta,\zeta) \equiv (\zeta+\eta)^2\sum_k(\zeta - x_k - \eta)^2\frac{x_k\Theta_k^2}{\Delta_k^2} \tag{8b}$$

$$F_z(\eta,\zeta) \equiv \sum_k [1+(\zeta+\eta)(\zeta - x_k - \eta)]\frac{\Theta_k^2}{\Delta_k} \tag{8c}$$

$$F_W(\eta,\zeta) = \sum_k \left\{[1+(\zeta+\eta)(\zeta - x_k - \eta)]^2 + x_k^2\right\}\frac{\Theta_k^2}{\Delta_k^2} \tag{8d}$$

where $\Delta_k = 1 + (\zeta - x_k)^2 - \eta^2$ and $x_k = l_i\Lambda_k = \gamma\sqrt{p_l^2 + (n\pi a/L)^2}$

This gives the desired form for the scaled ordered energy, R_W, as an implicit function of $\tan(\theta)$ and the scaled angular momentum parameter R_z:

$$\tan(\theta) = \frac{F_i}{F_e} \tag{9a}$$

$$R_z = F_z^2 \Big/ \sqrt{F_e^2 + F_i^2} \tag{9b}$$

$$R_W = F_W \Big/ \sqrt{F_e^2 + F_i^2} \tag{9c}$$

III. Treatment of Mathematical Pathologies

Inspection of equations 8a-d reveals that the sums become ill-defined at roots of the denominators Δ_k. However, the ratios of sums appearing in $\tan(\theta)$, R_z and R_W are well-defined and obtained by a limiting procedure. Figure 1 shows the root trajectories in ζ–η space for the first 240 roots. Along each curve in the figure, $\Delta_k = 0$ for one of the modes. All such roots are restricted to the region $|\eta| \geq 1$. Since equations (8a) -(9c) are symmetric under the substitution $\zeta \rightarrow -\zeta$ and $\eta \rightarrow -\eta$, we have restricted our considerations to the region $\eta \geq 0$ in Figure 1.

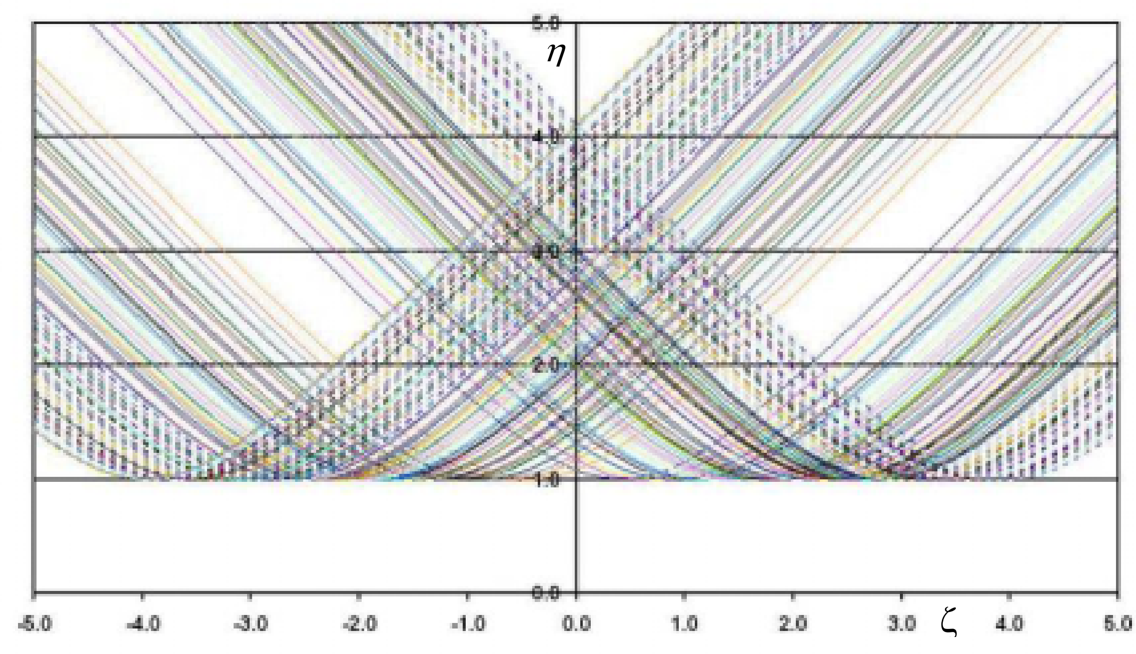

Figure 1 Root Trajectories for first 240 Eigenmodes

As indicated in Figure 1, we have three possibilities: 1. no roots (for $|\eta| \leq 1$ or in the spaces between the root trajectories); 2. a single mode, k_0, has a root; and 3. a pair of modes, k_0 and l_0, have roots. Each case must be treated differently, as follows.

Case 1: First, the helicity parameter $\tan(\theta)$ is selected and the corresponding $\tan(\theta)$ = constant trajectory in ζ–η space is obtained by a gradient-search method. Next, R_z and R_W are calculated along the trajectory. Finally, the resulting values of R_z and R_W, for fixed $\tan(\theta)$, are plotted. This process is repeated for various θ values, until the no-root region has been adequately canvassed to reveal the minimum ordered energy .

Case 2: This is the simplest case, as only a single mode contributes to each sum, resulting in the equations :

$$R_z \quad = \quad \frac{\left[1 \pm \sqrt{\tan(\theta)}\right]^2 \Theta_k^2}{|x_k|\sqrt{1+\tan(\theta)^2}} \qquad (10a)$$

$$R_W \quad = \quad \frac{x_k^2 + \left[1 \pm \sqrt{\tan(\theta)}\right]^2}{|x_k|\sqrt{1+\tan(\theta)^2}} \text{ where } 0 \leq \tan(\theta) < \infty \qquad (10b)$$

Case 3: In this case, two modes dominate as the double-singularity point is approached. In order to more easily deal with the double-mode roots, one final variable transformation is needed: $\mathrm{u} = \zeta + \sqrt{\eta^2 - 1}$; $\mathrm{v} = \zeta - \sqrt{\eta^2 - 1}$, which simplifies Δ_k to the form: Δ_k = (u-x_k)(v-x_k). By an extension of l'Hopital's theorem from one to two variables, and using the transformed variables u and v, one can show that equations (8a) through (9c) become:

$$\tan(\theta) = \frac{\tan(\alpha)^2 f^2 + g^2 R_0}{\tan(\alpha)^2 + R_0}; \; -\infty < \tan(\alpha) < \infty \tag{11a}$$

$$R_z = \frac{1}{\sqrt{1+\tan(\theta)^2}} \frac{|x_k|\Theta_k^2 \left[h\tan(\alpha) - h^{-1}R_0\right]^2}{\left|\tan(\alpha)^2 + R_0\right|} \tag{11b}$$

$$R_W = \frac{1}{\sqrt{1+\tan(\theta)^2}} \frac{|x_k|\left[\tan(\alpha)^2\{1+h^2\} + \{1+h^{-2}\}V_0\right]}{\left|\tan(\alpha)^2 + R_0\right|} \tag{11c}$$

where

$$f = 1 + \Lambda_k h; \; g = 1 + \Lambda_l h^{-1}; \; h = \sqrt{1+\Delta_0^2} + \Delta_0; \; \Delta_0 = \frac{1}{2}(x_k - x_l) > 0; \text{ and } R_0 = \frac{x_l \Theta_l^2}{x_k \Theta_k^2}; V_0 = \frac{x_l^2 \Theta_l^2}{x_k^2 \Theta_k^2}$$

and α is the angle at which the singularity is approached, relative to the u-axis. Note that we choose $x_k > x_l$ for both positive and negative eigenvalues.

IV. Results for Angular Momentum Conservation

Case 1: This region is defined by $-\infty < \zeta < \infty$ and $-1 \leq \eta \leq 1$, but symmetry under $\zeta \to -\zeta$ and $\eta \to -\eta$ means we may cover this region with the range $0 \leq \zeta < \infty$ and $-1 \leq \eta \leq 1$. Calculations so far have shown that the lowest ordered energy occurs for the helicity parameter θ = 65 degrees, at the boundary of the singular region, η = 1. In other words, it appears that the no-root region does **not** contain the global minimum, but rather that lower values for ordered energy are found in the root region.

Case 2: By comparing equations (10a, b) with the results of ref. 1 (in which angular momentum is not conserved), we see that the ordered energy for conserved L_z has the same functional form as the non-conserved case. Therefore, the single-mode root solutions yield a value for W_{MF} *no lower* than that obtained without angular momentum conservation.

Case 3: In Appendix A we show that the expressions for tan(θ), R_z and R_W at the two-mode singularity (equations (11a) through (11c)) yield the same functional form for normalized ordered energy as the non-conserved case described in ref. 1. Therefore, the double-mode root solutions yield a value for W_{MF} *no lower* than that obtained without angular momentum conservation.

Summary: Based on the results so far, conservation of angular momentum does **not** produce a lower ordered energy in the no-root region or at single- or double-mode singularities. The remaining possibility is shown in Figure 2, i.e., the interstices between the root trajectories. This is the final and most complex region in which to search, and is the object of our current investigations.

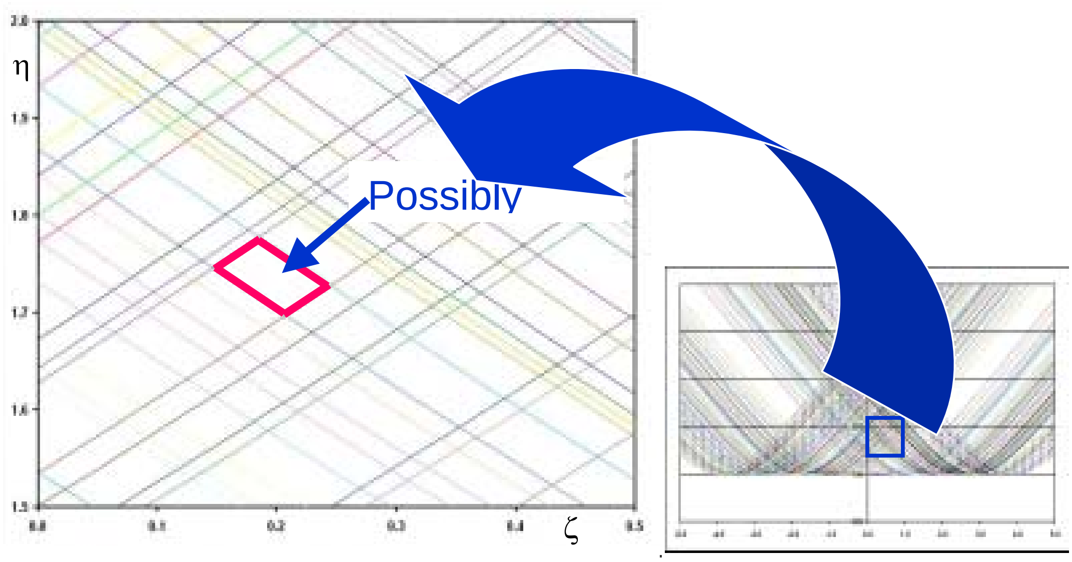

Figure 2 Remaining Possibility Location of Global Minimum

Appendix A Equivalence of Ordered Energy for Conserved and Non-conserved L_z at Double-Mode Singularities

From ref. 1, we have the equations for $\tan(\theta)$ and normalized W_{MF} :

$$\tan(\theta) = \frac{K_i}{K_e} = \frac{x_k f^2 A_k^2 + x_l g^2 A_l^2}{x_k A_k^2 + x_l A_l^2} = \frac{f^2\chi^2 + g^2 R}{\chi^2 + R};\ \chi = \frac{A_k}{A_l};\ -\infty < \chi < \infty \qquad \text{(13a)}$$

$$\frac{l_i W_{MF}}{|K|} = \frac{1}{\sqrt{1+\tan(\theta)^2}} \frac{x_k^2\left(1+h^2\right)A_k^2 + x_l^2\left(1+h^{-2}\right)A_l^2}{\left|x_k A_k^2 + x_l A_l^2\right|} = \frac{1}{\sqrt{1+\tan(\theta)^2}} \frac{|x_k|\left[\left(1+h^2\right)\chi^2 + \left(1+h^{-2}\right)R^2\right]}{\left|\chi^2 + R\right|} \qquad \text{(13b)}$$

where, as in (11a) through (11c):

$$f = 1 + x_k h;\ g = 1 + x_l h^{-1};\ h = \sqrt{1+\Delta_0^2} + \Delta_0;\ \Delta_0 = \frac{1}{2}(x_k - x_l) > 0;\ R = \frac{x_l}{x_k}$$

A_k and A_l = modal amplitudes; x_k and x_l = eigenvalues

To get equations (11a) through (11c) in the same form as (13a,b), note that $R_0 = R/\tau^2$ and $V_0 = R^2/\tau^2$, where $\tau = \Theta_k/\Theta_l$, so we obtain for conserved L_z:

$$\tan(\theta) \quad = \quad \frac{\tan(\alpha)^2 f^2 + g^2 R/\tau^2}{\tan(\alpha)^2 + R/\tau^2} = \frac{\{\tau\tan(\alpha)\}^2 f^2 + g^2 R}{\{\tau\tan(\alpha)\}^2 + R}; \; -\infty < \tau\tan(\alpha) < \infty \qquad (14a)$$

$$R_z = \frac{1}{\sqrt{1+\tan(\theta)^2}} \frac{|x_k|\Theta_k^2\left[h\tan(\alpha) - h^{-1}R/\tau^2\right]^2}{\left|\tan(\alpha)^2 + R/\tau^2\right|} = \frac{1}{\sqrt{1+\tan(\theta)^2}} \frac{|x_k|\Theta_k^2\left[h\tau\tan(\alpha) - h^{-1}R/\tau\right]^2}{\left|(\tau\tan(\alpha))^2 + R\right|} \qquad (14b)$$

$$R_W = \frac{1}{\sqrt{1+\tan(\theta)^2}} \frac{|x_k|\left[\tan(\alpha)^2\{1+h^2\} + \{1+h^{-2}\}R^2/\tau^2\right]}{\left|\tan(\alpha)^2 + R/\tau^2\right|} = \frac{1}{\sqrt{1+\tan(\theta)^2}} \frac{|x_k|\left[(\tau\tan(\alpha))^2\{1+h^2\} + \{1+h^{-2}\}R^2\right]}{\left|(\tau\tan(\alpha))^2 + R\right|} \qquad (14c)$$

Comparing (13b) with (14c) we see that the normalized ordered energy has the same dependence on $\tan(\theta)$ and the independent variables χ and $\tau\tan(\alpha)$ for both conserved and non-conserved angular momentum. Hence, the conservation of angular momentum does not yield a lower ordered energy at the double-mode singularities.

Reference:

[1] L.C. Steinhauer, Phys. Plasmas **9**, 3767 (2002).

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