Transient Behavior and Secularity on Interchange Instabilities in Shear Flow Plasmas

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Transient and secular behaviors of interchange fluctuations are analyzed in an ambient shear flow by invoking Kelvin's method of shearing modes. Because of its non-Hermitian property, complex transient phenomena can occur in a shear flow system. The combined effect of shear flow mixing and Alfvén wave propagation overcomes the instability driving force at sufficiently large time, and damps all fluctuations of the magnetic flux. On the other hand, electrostatic perturbations can be destabilized for sufficiently strong interchange drive. The time asymptotic behavior in each case is algebraic (non-exponential).

I. INTRODUCTION

It is widely accepted that a shear flow yields stabilizing effects on various fluctuations through convective deformations of disturbances [1, 2]. However, rigorous treatment of the shear flow effects encounters a fatal difficulty arising from the non-Hermitian (nonselfadjoint) properties of the problem. We may not consider well-defined "modes" and corresponding "time constants." The standard normal mode approach breaks down, and the theory may fail to give correct predictions of evolution even if perturbation fields remain in the linear regime. The discrepancies between the theory and the experiment on the stability limit of neutral fluids are reviewed in Ref. [3]. The aim of this work is to establish a solid foundation for the analysis of shear flow systems. We apply Kelvin's method of shearing modes [4]. This scheme, previously called as 'nonmodal' approach, actually consists in the combination of two methods which have been widely used in solving wave equations; the modal and the characteristics methods.

Much work has been done on instability problems with shear flows by means of the 'modal' approach. It is implicitly assumed in the application of the modal scheme that the motion can be decomposed into a set of independent normal 'modes' with certain time constants. As is well-known, this method is effective in solving problems involving Hermitian operators, however, when applying it to non-Hermitian systems, we may overlook the secular and transient behaviors. On the other hand, the characteristics method has been used in the context of rapid distortion theory for analyzing the fluid turbulence [5]. If we can treat the non-Hermitian part of the whole operator as a singular perturbation to a Hermitian operator [6], we may be able to construct the theory in the framework of the perturbation theory for the operator [7]. But unfortunately the convergence of the perturbative series seems to be very ambiguous in case of the shear flows due to the secularity of their time evolutions. Thus, a thorough mathematical treatment of the non-Hermitian properties of shear flow systems has not been accomplished so far. In this paper, we have analyzed the shear flow effect on interchange instabilities and its non-Hermitian mathematical background, deriving the time asymptotic behavior by means of Kelvin's method.

We will first revisit Kelvin's method from the viewpoint of the characteristics method (Sec. II). We will review the spectral theory focusing on the general mathematical concept of eigenmode in order to gain a better understanding of Kelvin's method. In Sec. III, we will formulate the equations for the interchange instabilities. In Sec. IV, we will derive the ordinary differential equation (ODE) in time for the evolution of the amplitude of the interchange instabilities by applying the analysis of shearing modes. In Sec. V, by drawing the analogy with Newton's equation it will be shown that the solution to the above mentioned ODE for the flux function exhibits an asymptotic damped behavior without any threshold of instability drive. We will summarize the result in Sec. VI.

II. NON-HERMITIAN PROPERTY OF SHEAR FLOW SYSTEMS

Before formulating the interchange instability equations, let us describe a rough sketch of the problem and explain the mathematical tool to analyze the non-Hermitian dynamics. The linearized dynamics of fluid systems in the presence of sheared flow is governed by a general equation of the following type;

$$\partial_t u + \boldsymbol{v} \cdot \nabla u = \mathcal{A} u, \tag{1}$$

where $\mathcal{A}$ denotes a Hermitian differential operator (time-independent) defined in a Hilbert space V, $\boldsymbol{v}$ is the stationary mean flow, and u ($\in V$) denotes a perturbation field.

It is the convective derivative, $\boldsymbol{v} \cdot \nabla$, that introduces the non-Hermitian property into problem (1) and prevents the possibility of representing the dynamics of the systems in terms of orthogonal and complete set of eigenfunctions. This is a well known difficulty in the stability analysis of neutral fluids, such as Couette or Poiseille flows, where the predictions obtained by means of the modal methods do not match the experiments [3].

In the case of a spatially inhomogeneous stationary flow $\boldsymbol{v}$, Eq. (1) becomes non-Hermitian and a straightforward spectral resolution is not effective. However, Kelvin's method permits to resolve, for some classes of mean flows, the evolution of

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the system (1) into new types of modes by means of which both transient and secular asymptotic behaviors are effectively described. Let us now explain the mathematical foundations of this scheme.

As mentioned in Sec. I, Kelvin's method consists in the combined application of two methods which have been extensively used in the analysis of wave equations. Precisely the "Lagrangian" part of Eq. (1), $\partial_t + \boldsymbol{v} \cdot \nabla$, is solved by means of the characteristics method and the "Hermitian" part $\mathcal{A}$ by means of the standard spectral resolution.

The characteristics method is applied to solve the characteristic ODE associated to the Lagrangian derivative moving along the characteristic curve of the ambient motion, which is given by

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{v}, \quad \boldsymbol{x}(0) = \boldsymbol{\xi} \tag{2}$$

By inverting the modes, which are expressed in Lagrangian coordinates as $\varphi(\boldsymbol{k}, \boldsymbol{\xi})$, they will be represented in Eulerian coordinates as

$$\tilde{\varphi}(t; \boldsymbol{k}, \boldsymbol{x}) = \varphi(\boldsymbol{k}, \boldsymbol{\xi}(t; \boldsymbol{x})), \tag{3}$$

where $\boldsymbol{\xi}(t; \boldsymbol{x})$ denotes the inverse of $\boldsymbol{x}(t; \boldsymbol{\xi})$. The existence of the inverse mapping $\boldsymbol{x}(t) \mapsto \boldsymbol{\xi}$ is guaranteed in the case of incompressible mean flows. Due to Eq. (3), $\tilde{\varphi}(t; \boldsymbol{k}, \boldsymbol{x})$ satisfies the characteristic equation

$$\partial_t \tilde{\varphi}(t; \boldsymbol{k}, \boldsymbol{x}) + \boldsymbol{v} \cdot \nabla \tilde{\varphi}(t; \boldsymbol{k}, \boldsymbol{x}) = 0. \tag{4}$$

The essential condition for the applicability of Kelvin's method consists in the constraint for the functions $\tilde{\varphi}(t; \boldsymbol{k}, \boldsymbol{x})$ to form the complete set of eigenfunctions of the operator $\mathcal{A}$. If such a set of eigenfunctions exists, we can decompose the perturbation field u by means of

$$u = \int \hat{u}_k(t)\, \tilde{\varphi}(t; \boldsymbol{k}, \boldsymbol{x})\, \mathrm{d}\boldsymbol{k}. \tag{5}$$

We notice that due to Eq. (3) the eigenvalues of $\mathcal{A}$ become time dependent. The new eigenvalue problem for $\mathcal{A}$ reads

$$\mathcal{A}\tilde{\varphi}(t; \boldsymbol{k}, \boldsymbol{x}) = \lambda_k(t)\, \tilde{\varphi}(t; \boldsymbol{k}, \boldsymbol{x}). \tag{6}$$

Plugging Eq. (5) into Eq. (1) and exploiting Eqs. (4) and (6), we have

$$\int [\partial_t \hat{u}_k(t)]\, \tilde{\varphi}(t; \boldsymbol{k}, \boldsymbol{x})\, \mathrm{d}\boldsymbol{k} = \int \hat{u}_k(t) \lambda_k(t)\, \tilde{\varphi}(t; \boldsymbol{k}, \boldsymbol{x})\, \mathrm{d}\boldsymbol{k}. \tag{7}$$

Due to the orthogonality of the modes $\tilde{\varphi}(t; \boldsymbol{k}, \boldsymbol{x})$, the evolution of $\hat{u}_k$ is governed by the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{u}_k(t) = \lambda_k(t)\, \hat{u}_k(t). \tag{8}$$

If $\tilde{\varphi}(t; \boldsymbol{k}, \boldsymbol{x})$ do not satisfy both conditions given by characteristic equation (4) and eigenequation (6), Eq. (7) will have additional terms which represent the complicated mode coupling and thus the applicability of Kelvin's method is compromised.

Due to the time dependence present in the eigenvalues $\lambda_k(t)$, the evolution of $\hat{u}_k(t)$ will not exhibit a simple exponential dependence as in the Hermitian case, but more complicated behaviors, which are characteristic of non-Hermitian systems. By analyzing this ODE, the motion of each mode can be classified, and the time asymptotic behavior can be also shown. The following sections will be devoted to the derivation of ODE (8) and the discussion of the behavior of its solution for interchange instabilities in plasmas with shear flow.

Finally, let us close this section by presenting the following theorem. The proof is shown in Ref. [8]

Theorem II.1 Let us consider in an unbounded domain the system of PDEs

$$(\partial_t + \boldsymbol{A}(\boldsymbol{x}) \cdot \nabla)[\mathcal{L}(t)\boldsymbol{\psi}] = \mathcal{H}(t)\boldsymbol{\psi} \tag{9}$$

where

1. $\boldsymbol{A}(\boldsymbol{x})$ is an n-dimensional vector function linear in the components of $\boldsymbol{x} = (x_1, \ldots, x_n)$
2. $\mathcal{L}(t, \partial_{x_1}, \ldots, \partial_{x_n})$ and $\mathcal{H}(t, \partial_{x_1}, \ldots, \partial_{x_n})$ are linear differential matrix ($m \times m$) operators, depending explicitly on time, which act on a m-dimensional vector function $\boldsymbol{\psi}$

Then the solutions of Eq. (9) obtained by means of Kelvin's method are general.

III. FORMULATION OF INTERCHANGE INSTABILITIES

In this section, we will derive the equations for stationary flowing plasmas. Specifically we will investigate the effect of shear flows on interchange instabilities of plasma under the influence of homogeneous magnetic field.

In the presence of gravitational force, the ideal MHD equations read as

$$\rho\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \boldsymbol{j} \times \boldsymbol{B} - \nabla p + \rho\boldsymbol{g}, \tag{10}$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho\nabla \cdot \boldsymbol{v} = 0, \tag{11}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} - \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) = 0, \tag{12}$$

$$\nabla \cdot \boldsymbol{v} = 0, \tag{13}$$

where ρ, $\boldsymbol{B}$, and $\boldsymbol{g}$ are the density, magnetic field, and gravitational constant vector, and $\mathrm{d}/\mathrm{d}t = \partial_t + \boldsymbol{v} \cdot \nabla$ denotes the Lagrangian derivative. Here we assume the incompressibility of the velocity field $\boldsymbol{v}$, instead of using the equation of state.

The ambient fields (denoted by the subscript 0) must satisfy

$$\rho_0 \boldsymbol{v}_0 \cdot \nabla \boldsymbol{v}_0 = \boldsymbol{j}_0 \times \boldsymbol{B}_0 - \nabla p_0 + \rho_0 \boldsymbol{g}. \tag{14}$$

If we consider a parallel stationary shear flow of the form $\boldsymbol{v}_0 = (0, v_{0y}(x), 0)$, straight homogeneous magnetic field

$\boldsymbol{B}_0 = (0, B_y, B_z)$, and gravitational force acting in the positive x direction, the convective derivative gives no contribution to the stationary state and Eq. (14) is reduced to

$$\nabla p_0 = \rho_0 \boldsymbol{g}. \tag{15}$$

The perturbed magnetic and velocity fields are assumed to be two dimensional in x-y plane, and thus we can introduce the poloidal flux function and stream function;

$$\begin{aligned} \boldsymbol{B}_{1\perp} &= \nabla\psi \times \boldsymbol{e}_z, \\ \boldsymbol{v}_{1\perp} &= \nabla\phi \times \boldsymbol{e}_z, \end{aligned} \tag{16}$$

where the subscript 1 denotes the perturbed quantities, $\perp$ expresses the direction perpendicular to the z axis, and $\boldsymbol{e}_z$ denotes the unit vector in the z direction.

Taking the curl of the equation of motion and projecting it along $\boldsymbol{e}_z$, we obtain

$$\mu_0\rho_0[(\partial_t + v_{0y}\partial_y)\nabla_\perp^2\phi - v_{0y}''\partial_y\phi] = \boldsymbol{B}_0\cdot\nabla(\nabla_\perp^2\psi) + \mu_0\partial_y\rho_1 g, \tag{17}$$

where $\nabla_\perp^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$. In deriving Eq. (17) we have used the Boussinesq approximation which consists in the neglection of the spatial variation of the stationary state density in the inertial term of equation of motion, but not in continuity equation since it is the driving term for the interchange instability. The component of the flow perpendicular to the ambient magnetic field can be considered consistently coming from the $\boldsymbol{E}\times\boldsymbol{B}$ drift, taking into account the ideal Ohm's law. It is noted that, if we neglect the effect of the magnetic field, we recover the Rayleigh's equation for Kelvin-Helmholtz instability [9].

The density fluctuation can be expressed as

$$(\partial_t + v_{0y}\partial_y)\rho_1 = -\rho_0'\partial_y\phi. \tag{18}$$

where the prime denotes the derivative with respect to x. Now ρ_0' is considered as a constant which introduces a destabilizing force. The induction equation is the same as in the ordinary reduced MHD equations [10] and under the above assumptions on the stationary fields reads as

$$(\partial_t + v_{0y}\partial_y)\psi = \boldsymbol{B}_0\cdot\nabla\phi. \tag{19}$$

Equations (17)-(19) constitute a closed system of equations. We can see that the static system ($v_{0y} = 0$) governed by these equations shows Hermitian property, and the convective derivative ($v_{0y} \neq 0$) brings the non-Hermitian property into our system. It sould be also noted that this system of equations is equivalent to RMHD equations for tokamaks of high beta ordering [10]. We will investigate the effect of the shear flow on the interchange instabilities in following sections.

IV. DERIVATION OF ORDINARY DIFFERENTIAL EQUATION

In this section, we derive the ODE for the amplitude of Kelvin's modes, given in Eq. (8), in the case of interchange instabilities of plasmas. Let us consider the electromagnetic

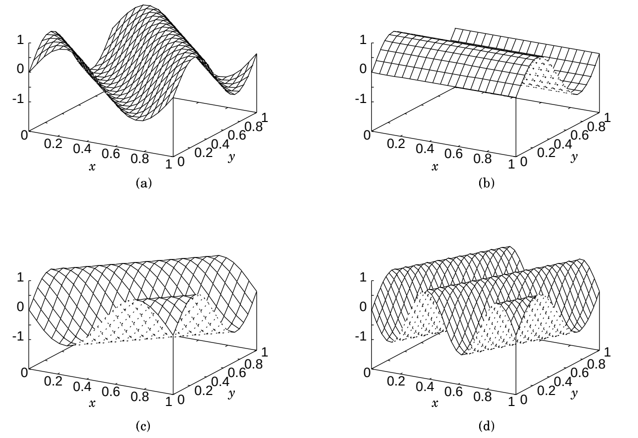

FIG. 1: Kelvin's shearing mode $\tilde{\varphi}(t;\boldsymbol{k},\boldsymbol{x})$. The mode is being stretched due to the y directed flow which is sheared in x direction.

case where $\boldsymbol{B}_0\cdot\nabla \neq 0$. The analysis for electrostatic case ($\boldsymbol{k}\cdot\boldsymbol{B}_0 = 0$) is shown in Ref. [11]. ¿From Eqs. (18)-(19), we have

$$\phi = -\partial_y^{-1}\rho_0'^{-1}(\partial_t + v_{0y}\partial_y)\rho_1 = (\boldsymbol{B}_0\cdot\nabla)^{-1}(\partial_t + v_{0y}\partial_y)\psi. \tag{20}$$

Since we have assumed the mean velocity $v_{0y} = v_{0y}(x)$ and the homogeneous ambient field $\boldsymbol{B}_0 = (0, B_y, B_z)$, the operator $\partial_t + v_{0y}\partial_y$ commutes with both ∂_y^{-1} and $(\boldsymbol{B}_0\cdot\nabla)^{-1}$. Thus acting on both sides of Eq. (20) with the operator $(\partial_t + v_{0y}\partial_y)^{-1}$ gives

$$\rho_1 = -\rho_0'\partial_y(\boldsymbol{B}_0\cdot\nabla)^{-1}\psi. \tag{21}$$

¿From Eq. (19),

$$\nabla_\perp^2\phi = \nabla_\perp^2(\boldsymbol{B}_0\cdot\nabla)^{-1}(\partial_t + v_{0y}\partial_y)\psi. \tag{22}$$

Substituting Eqs. (20) and (22) into Eq. (17), and acting with $\boldsymbol{B}_0\cdot\nabla$ on both sides, we obtain

$$(\partial_t + v_{0y}\partial_y)\nabla_\perp^2(\partial_t + v_{0y}\partial_y)\psi = \frac{(\boldsymbol{B}_0\cdot\nabla)^2}{\mu_0\rho_0}\nabla_\perp^2\psi - \frac{\rho_0' g}{\rho_0}\partial_y^2\psi. \tag{23}$$

Since the operator on the right hand side is Hermitian, we can decompose the flux function ψ by means of the shearing eigenmodes

$$\psi(\boldsymbol{x}, t) = \int \hat{\psi}_k(t)\,\tilde{\varphi}(t;\boldsymbol{k},\boldsymbol{x})\,\mathrm{d}\boldsymbol{k}, \tag{24}$$

where each eigenmode can be expressed by the sinusoidal function in our simplified case

$$\begin{aligned} \tilde{\varphi}(t;\boldsymbol{k},\boldsymbol{x}) &= \exp[\mathrm{i}k_x x + \mathrm{i}k_y(y - v_{0y}t) + \mathrm{i}k_z z] \\ &= \exp[\mathrm{i}\tilde{k}_x(t)x + \mathrm{i}k_y y + \mathrm{i}k_z z]. \end{aligned} \tag{25}$$

Here the mean flow is assumed to be $v_{0y}(x) = \sigma x$ and $\tilde{k}_x(t) = k_x - k_y\sigma t$. It is explicitly shown that the wave number in the flow shear direction is linearly increasing with time due to the distorting effect of the mean flow. Since continuous variation of $\tilde{k}_x(t)$ prevents from imposing the boundary condition in the bounded domain, we will concentrate on the analysis of localized perturbations by considering the infinite domain. Note that $\tilde{\varphi}$ are the eigenfunctions of the right hand side of Eq. (23), and also satisfy the characteristic equation (4). It should be noted that the presence of the Laplacian operator in the left hand side of Eq. (23) does not hinder the application of Kelvin's method since the modes $\tilde{\varphi}$ are as well eigenfunctions of the Laplacian $\nabla_\perp^2$.

Thus, the time evolution equation for the amplitude $\hat{\psi}_k$ can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\tilde{k}_x(t)^2+k_y^2\right)\frac{\mathrm{d}\hat{\psi}}{\mathrm{d}t}\right]=-\frac{F^2}{\mu_0\rho_0}\left(\tilde{k}_x(t)^2+k_y^2\right)\hat{\psi}-k_y^2\frac{\rho_0' g}{\rho_0}\hat{\psi}, \tag{26}$$

where $F=\boldsymbol{k}\cdot\boldsymbol{B}_0=k_yB_{0y}+k_zB_{0z}$, and we have dropped the subscript k for simplicity. We notice that in the absence of shear flow ($\sigma=0$) the usual interchange instability equation for static equilibrium can be obtained.

Normalizing the time t by the poloidal Alfvén time $\tau_A = a\sqrt{\mu_0\rho_0}/F$, we can rewrite Eq. (26) in dimensionless form as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\tilde{k}_x(t)^2+k_y^2\right)\frac{\mathrm{d}\hat{\psi}}{\mathrm{d}t}\right]=-\left(\tilde{k}_x(t)^2+k_y^2\right)\hat{\psi}+k_y^2\frac{\tau_A^2}{\tau_G^2}\hat{\psi}, \tag{27}$$

where the wave vectors are normalized by the characteristic length scale a and $\tau_G^2=-\rho_0/\rho_0' g$. Further we can rewrite Eq. (27) in the form

$$\frac{\mathrm{d}^2\hat{\psi}}{\mathrm{d}t^2}+\mu(t)\frac{\mathrm{d}\hat{\psi}}{\mathrm{d}t}+[1-S(t)]\hat{\psi}=0, \tag{28}$$

where

$$\mu(t)=-\frac{2\sigma k_y\tilde{k}_x(t)}{\tilde{k}_x(t)^2+k_y^2},$$
$$S(t)=\frac{k_y^2G}{\tilde{k}_x(t)^2+k_y^2},$$

and $G=\tau_A^2/\tau_G^2$. Drawing an analogy with Newton's equation, $\mu(t)$ represents the frictional term and $S(t)$ the interchange drive term. Equation (28) is the correspondent of Eq. (8). As we have mentioned in Sec. II, the time evolution for the amplitude of each eigenmode is no longer a simple exponential function. The behavior of $\hat{\psi}$ will be discussed in the following sections.

V. ASYMPTOTIC AND TRANSIENT BEHAVIOR OF EACH MODE

In the absence of a density gradient or shear flow, $\mu(t)=S(t)=0$ in Eq. (28) and we have a pure oscillation representing the Alfvén wave. If we include the density gradient, then $S(t)\neq 0$ and we obtain the interchange instability for negative ρ_0'. Since a homogeneous magnetic field is assumed in this paper, we have no stabilizing effect of the magnetic shear. The operator is Hermitian in these two cases, therefore we have the simple exponential evolution with time constants for each mode.

Case	$\sigma k_x k_y$	$\mu(t=0)$	$\mu(t\to\infty)$
(a)	$-$	$+$	$+$
(b)	$+$	$-$	$+$

TABLE I: Classification of the signs of parameter product $\sigma k_x k_y$ and effective frictional coefficient $\mu(t)$.

When we include the shear flow, we have $\mu(t)\neq 0$ and we can draw an analogy with the dynamics of a damped oscillator with time dependent frictional coefficient $\mu(t)$. When time goes, $\mu(t)$ becomes always positive, which means a formal dissipation, and therefore the oscillation energy of the Alfvén wave $[(\mathrm{d}\hat{\psi}/\mathrm{d}t)^2+\hat{\psi}^2]/2$ decreases monotonically. In the following subsections we will describe both the asymptotic and transient behaviors of the amplitudes $\hat{\psi}$.

A. Transient behavior

Let us first look the transient behavior of each mode. Since an analytic expression is not available, we discuss the transients by qualitatively analyzing the ODE (28). In the absence of the instability drive, we will have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\frac{\mathrm{d}\hat{\psi}}{\mathrm{d}t}\right)^2+\hat{\psi}^2\right]=-\mu(t)\left(\frac{\mathrm{d}\hat{\psi}}{\mathrm{d}t}\right)^2, \tag{29}$$

where

$$\mu(t)=-\frac{2\sigma k_y\tilde{k}_x(t)}{\tilde{k}_x(t)^2+k_y^2},$$
$$\tilde{k}_x(t)=k_x-\sigma k_y t.$$

Therefore, the frictional coefficient $\mu(t)$ acts as a damping force for $\mu>0$. Since the sign of the denominator in $\mu(t)$ is always positive, the behavior will be determined by that of the numerator. The numerator can be expressed as $2\sigma^2k_y^2t-2\sigma k_yk_x$ and according to its initial value we can single out two classes of the transients. When the product σk_yk_x is negative [TABLE I(a)], the frictional coefficient $\mu(t)$ is always positive from the beginning, therefore the shear flow acts as a damping force at any time and the mode shows simple damped behavior. On the other hand, if the product σk_yk_x is positive [TABLE I(b)], the frictional coefficient $\mu(t)$ is initially negative and changes its sign at the instant $t_*=k_x/\sigma k_y$. Therefore the mode experiences an initial amplification lasting until the time t_*, which is even faster than it would be in the presence of the only interchange drive. This transient behavior can also be seen by numerically integrating Eq. (28) which is shown in Fig. 3, where the initial amplification lasts till the turning point $t_*=50$ followed by the asymptotic decaying phase.

We have observed by numerical integration that the amplitude can be amplified to values of 10^{30} times larger than the

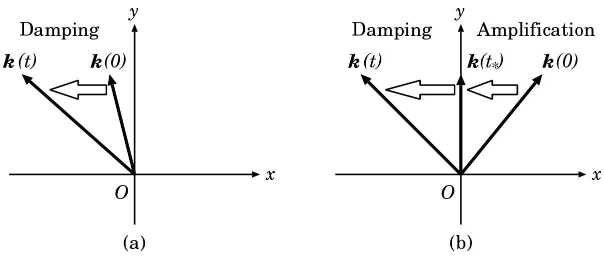

FIG. 2: Time evolution of the wave vector $\boldsymbol{k}$. We have taken $\sigma > 0$, $k_y > 0$.

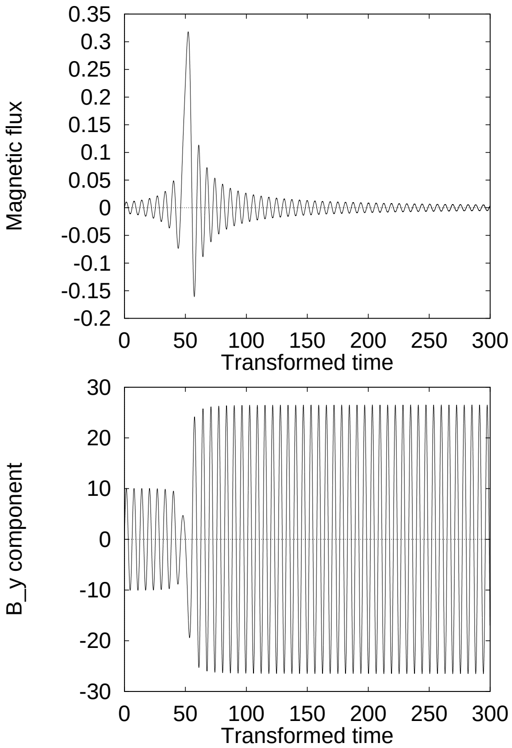

FIG. 3: Direct numerical integration of Eq. (28) for each mode. The parameters are follows: $k_x = 10$, $k_y = 1$, $k_z = 0$, $\sigma = 0.2$, $G = 1$, and initial perturbations $\hat{\psi} = 0$ and $\mathrm{d}\hat{\psi}/\mathrm{d}t = 1.0$ at $t = 0$.

initial one. ¿From a physical point of view, such huge amplifications may break down the linearity of the perturbations and may lead to a nonlinear stage. This case is beyond the scope of the linear theory and no sure conclusion can be drawn from Kelvin's method. Such huge amplifications are experienced by modes with large t_* and G.

B. Asymptotic behavior

In order to study the time asymptotic behavior, we assume $t \gg k_x/\sigma k_y$, $1/\sigma$. In this time asymptotic limit we obtain the following ODE

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\hat{\psi} + \frac{2}{t}\frac{\mathrm{d}}{\mathrm{d}t}\hat{\psi} + \left(1 - \frac{G/\sigma^2}{t^2}\right)\hat{\psi} = 0, \tag{30}$$

where $G = \tau_A^2/\tau_G^2$ denotes the magnitude of the instability drive term. In the absence of the instability drive G, the time asymptotic behavior of the solution of Eq. (30) is expressed as

$$\hat{\psi} \sim \frac{1}{t}\sin t, \tag{31}$$

which coincides with the result of Koppel [12] which considered a time dependent non-perturbative state. Since Eq. (30) is the spherical Bessel equation, its general solution for $G \neq 0$ is expressed by

$$\hat{\psi} = \frac{1}{\sqrt{t}}\big(C_1 J_\nu(t) + C_2 Y_\nu(t)\big), \tag{32}$$

where J_ν and Y_ν denote the Bessel functions, and $\nu = (G/\sigma^2 + 1/4)^{1/2}$. Therefore the time asymptotic behavior of the mode is expressed generally as

$$\hat{\psi} \sim \frac{1}{t}\sin\left(t - \frac{\pi\nu}{2} + \delta\right), \tag{33}$$

where δ denotes a constant phase depending on the initial values. Therefore the mode oscillates with amplitude $\hat{\psi}$ decaying with the inverse power of time. While the x component of the perturbation magnetic field $\hat{b}_x$ is proportional to ψ, the y component $\hat{b}_y$ is proportional to $\tilde{k}_x(t)\hat{\psi}$. Thus $\hat{b}_y$ tends to the pure oscillatory behavior

$$\hat{b}_y \sim \sin\left(t - \frac{\pi\nu}{2} + \delta\right), \tag{34}$$

as $\tilde{k}_x(t)$ increases with proportional to time (see Fig. 3). It should be noted that there is no threshold value for the stabilization of the interchange instability, since we obtain the same spherical Bessel equation (30) for all modes. All modes evolve as in Eq. (30) independently of the values of wave numbers $\boldsymbol{k}$.

The final amplitude of each mode depends sensitively on the parameters. As the shear parameter increases, the final amplitude of $\hat{b}_y$ tends to be larger as is also shown by Chagelishvili *et al*. [13], while the mixing damping effect on $\hat{b}_x$ increases. It should be noted that the instability drive G asymptotically has the only effect to shift the phase of the oscillations as can be seen in Eqs. (33) and (34), and it does not affect the principal time dependence. The combined effect of the Alfvén wave propagation and shear flow mixing always overcomes the interchange drive and the oscillations of the magnetic flux asymptotically decay with proportional to the inverse power of time.

VI. SUMMARY

Kelvin's method of shearing modes is interpreted as a combination of modal and characteristic methods for the analysis of a non-Hermitian system. Moreover, the importance of this analysis is that, Kelvin's method gives the general solution for specific shear flow problem which carries the non-Hermitian operator. Physically, Kelvin's mode shows that a shear flow distorts each Fourier mode, resulting in change of the wave number, which represents the stretching effect of the shear flow.

By means of this method, we have analyzed the incompressible electromagnetic perturbations in the presence of an interchange drive and obtained the ordinary differential equation (28) for the amplitude of the modes $\hat{\psi}_k$. All modes show asymptotic decay proportional to the inverse power of time (non-exponential) without any threshold value. This means that the interchange instabilities are always damped away at sufficiently large time due to the combined effect of the Alfvén wave propagation and distortion of modes by means of the background shear flow; i.e. phase mixing effect. However, the transient behavior is not common for all modes. Depending on the initial wave number, some of them show transient amplifications which are even faster than they would be in the presence of the only interchange drive. These amplifications are so conspicuous that they may lead to the break down of the linearity of the perturbation fields. Moreover, since the wave number increases linearly with respect to time, viscousity or resistivity may act most strongly if we include them. But the most important fact in this analysis is that, the mixing effect of the shear flow is stronger than destabilization effect due to gravitation, and the latter effect does not seriously act asymptotically.

It should be noted that, since our treatment considers the case of parallel linear shear flow, Kelvin-Helmholtz instabilities, which originate from the second order spatial derivative of the background shear flow [9, 14], are beyond the scope of the present theory. ¿From a mathematical point of view, we stress that the Kelvin-Helmholtz instability is a problem involving purely non-Hermitian operators in the sense that the operator $\mathcal{A}$ of Eq. (1) itself becomes non-Hermitian and therefore the method developed in Sec. II cannot be applied. This is a well known instability in fluid dynamics whose rigorous mathematical treatment presents highly non-trivial difficulties.

We note that the ODE which gives the evolution of the amplitudes of the interchange modes (28) and that of kink-type modes (Eq. (32) in Ref. [15]) are mathematically equivalent. Of course these two modes may have spatially different structures, at least this is the case for static equilibria. But this fact means that they have no difference in time evolution, and we can say that these terms have the same effect in the sense that they enlarge the spectrum to unstable eigenvalues. This equivalence stems from the assumption of a spatially homogeneous magnetic field.

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