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Stability Analysis of Flowing Two-Fluid Plasmas

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I. INTRODUCTION

It has been recognized that a shear flow may have a strong effect on transport and stability. However, the modes analyzed have largely been limited to electrostatic [1] and ballooning modes [2], and only the toroidal equilibrium flow has been taken into account. For high beta plasmas, the effect of flow on ~~a~~ global mode stability has received little attention although recent measurements on neutral beam driven STs may indicate significant poloidal and toroidal sheared flow [3]. Global modes of a Z-pinch with axial flow was investigated in Refs. [4] and [5] using a spectral treatment of the flowing one-fluid MHD model. This is the same plasma model considered by Frieman-Rotenberg [6]. A completely different approach, the theory of minimum energy states for a two species magnetofluid predicts that the relaxed states have large poloidal and toroidal flows with significant pressure gradients [7].

For detailed study of the effect of flow on the global mode stability of the high beta plasmas, it is useful to develop a simple model capable of comparing the stability of *static* and *flowing* equilibria. We present here a new formalism of *stability analysis of flowing two-fluid plasmas*. The basic equations are described in Sec.2. To simplify the analysis, we assume constant density. In Sec.3, the coupled equations for 2-D axisymmetric equilibria with both toroidal and poloidal flows are reviewed. The stability formalism is presented in Sec.4. Here a new relation between the perturbed magnetic field and the ion displacement is derived. Important differences between two-fluid and one-fluid plasmas are shown. In Sec. 5 results are shown, and Sec.6 summarizes the paper and presents a plan for future work.

II. BASIC EQUATIONS

It is assumed that plasma consists of an ion fluid and an electron fluid. To simplify the analysis, we also assume that the density is constant in time and uniform in space ($n_i \cong n_e \equiv n = \text{const.}$). The continuity equations imply that the each fluid species is incompressible. The basic equations that govern the system are given by

$$\nabla \bullet \mathbf{u}_\alpha = 0 \tag{2.1}$$

$$m_\alpha n\left[\frac{\partial \mathbf{u}_\alpha}{\partial t} + \mathbf{u}_\alpha \bullet \nabla \mathbf{u}_\alpha\right] = -\nabla p_\alpha + q_\alpha n\left(\mathbf{E} + \frac{1}{c}\mathbf{u}_\alpha \times \mathbf{B}\right) \tag{2.2}$$

$$\nabla \times \mathbf{B} = \frac{4\pi e n}{c}\left(\mathbf{u}_i - \mathbf{u}_e\right) \tag{2.3}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \tag{2.4}$$

$$\nabla \bullet \mathbf{B} = 0 \tag{2.5}$$

where m_α, q_α, $\mathbf{u}_\alpha$, p_α, $\mathbf{E}$ and $\mathbf{B}$ are the mass, charge, velocity, and pressure of a fluid species, the electric field and the magnetic field, respectively. The subscript $\alpha = i$, e denotes the species, and c is the speed of light. The displacement current is neglected in (2.3). In the following the electron mass is neglected because attention is limited to the stability of low frequency modes. In the following, the subscript 0 (1) e.g. $\mathbf{u}_{i0}$ ($\mathbf{u}_{i1}$) denotes the equilibrium (perturbed) quantity.

III. AXISYMMETRIC EQUILIBRIUM

The general formulation of axisymmetric two-fluid equilibria was presented in Ref. [8]. Assuming also uniform density the equilibrium magnetic field and flow can be expressed using the magnetic flux function $\psi(r,z)$ and the ion stream function $\psi_i(r,z)$ as

$$\mathbf{B}_0(r,z) = \frac{1}{r}\nabla\psi(r,z)\times\hat{\boldsymbol{\theta}} + B_{0\theta}(r,z)\hat{\boldsymbol{\theta}} \qquad (3.1)$$

$$\mathbf{u}_{i0}(r,z) = \frac{1}{r}\nabla\psi_i(r,z)\times\hat{\boldsymbol{\theta}} + u_{i0\theta}(r,z)\hat{\boldsymbol{\theta}} \qquad (3.2)$$

where $\hat{\boldsymbol{\theta}}$ is the unit vector in the azimuthal direction and cylindrical coordinates (r,θ,z) are used. Then, the following algebraic relations follow [8].

$$\frac{p_{e0}}{n} - e\phi_{E0} = H_{e0}(\psi) \quad (3.3), \qquad rB_{0\theta} - \frac{4\pi en}{c}\psi_i = \Phi(\psi) \quad (3.4)$$

$$\frac{p_{i0}}{n} + \frac{m_i}{2}u_{i0}^2 + e\phi_{E0} = H_{i0}(\psi_i) \quad (3.5), \qquad ru_{i0\theta} + \frac{e}{m_i c}\psi = \Phi_i(\psi_i) \quad (3.6)$$

where ϕ_{E0} is the electrostatic potential, $\Phi(\psi)$ and $H_e(\psi)$ are arbitrary functions of ψ, and $\Phi_i(\psi_i)$ and $H_i(\psi_i)$ are arbitrary functions of ψ_i. The functions $H_e(\psi)$ and $H_i(\psi_i)$ are *total* enthalpies (per particle) of each fluid species. Using these expressions the equilibrium equations reduce to a coupled system for ψ and ψ_i,

$$\Delta^*\psi + \Phi'\left(\Phi + \frac{4\pi en}{c}\psi_i\right) + \frac{4\pi en}{c}\left(\Phi_i - \frac{e}{m_i c}\psi\right) + 4\pi nr^2 H_e' = 0 \qquad (3.7)$$

$$\Delta^*\psi_i + \Phi_i'\left(\Phi_i - \frac{e}{m_i c}\psi\right) - \frac{e}{m_i c}\left(\Phi + \frac{4\pi en}{c}\psi_i\right) - \frac{1}{m_i}r^2 H_i' = 0 \qquad (3.8)$$

where Δ^* is the Grad-Shafranov operator, and the prime denotes the derivative of a function, e.g. $\Phi' = d\Phi(\psi)/d\psi$.

Note that the coupled system has four arbitrary functions. By virtue of the incompressibility assumption, this system is much simpler than the more general one given in Ref.[8]. In the absence of flow, (3.7) reduces to the familiar Grad-Shafranov equation.

IV. STABILITY ANALYSIS

Introduce the Lagrange displacement vector of the *ion* fluid $\boldsymbol{\xi}$ as in Ref. [9]

$$\mathbf{u}_{i1} = \frac{\partial\boldsymbol{\xi}}{\partial t} + \mathbf{u}_{i0}\bullet\nabla\boldsymbol{\xi} - \boldsymbol{\xi}\bullet\nabla\mathbf{u}_{i0} \qquad (4.1)$$

Then, the incompressibility implies

$$\nabla\bullet\boldsymbol{\xi} = 0 \qquad (4.2)$$

Taking the curl of the perturbed equation of motion of the ion fluid leads shortly to

$$\boldsymbol{\Omega}_1 = \nabla\times(\boldsymbol{\xi}\times\boldsymbol{\Omega}_0) \qquad (4.3)$$

where $\boldsymbol{\Omega}_0 = m_i\nabla\times\mathbf{u}_{i0} + \frac{e}{c}\mathbf{B}_0$ is the equilibrium generalized vorticity of the ion fluid [7] and $\boldsymbol{\Omega}_1$ is its perturbation. The relation (4.3) is natural extension of the one-fluid model [6] where $\boldsymbol{\Omega}_1$ takes the place of $\mathbf{B}_1$ and $\boldsymbol{\Omega}_0$ takes the place of $\mathbf{B}_0$. This generalization is one of the most important results of the two fluid analysis.

Taking the sum of the perturbed equations of motion of the ion and electron fluids leads to

$$\mathbf{f}(\boldsymbol{\xi}) - \nabla\left(p_1 + \frac{1}{4\pi}\mathbf{B}_0 \bullet \mathbf{B}_1\right) = 0 \qquad (4.4)$$

where the sum of the two perturbation pressures is $p_1 \equiv p_{i1} + p_{e1}$ and

$$\begin{aligned}\mathbf{f}(\boldsymbol{\xi}) \equiv & -n m_i \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} - 2 m_i n (\mathbf{u}_{i0} \bullet \nabla)\frac{\partial \boldsymbol{\xi}}{\partial t} + \\ & + \nabla \bullet [\, m_i n \boldsymbol{\xi} \{(\mathbf{u}_{i0} \bullet \nabla)\mathbf{u}_{i0}\} - m_i n \mathbf{u}_{i0} \{(\mathbf{u}_{i0} \bullet \nabla)\boldsymbol{\xi}\}\,] + \\ & + \frac{1}{4\pi}[(\mathbf{B}_0 \bullet \nabla)\mathbf{B}_1 + (\mathbf{B}_1 \bullet \nabla)\mathbf{B}_0]\end{aligned} \qquad (4.5)$$

and from (4.3) the perturbed magnetic field $\mathbf{B}_1$ is expressed as,

$$\mathbf{B}_1 = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0) - \frac{m_i c}{e}\nabla \times \frac{\partial \boldsymbol{\xi}}{\partial t} - \frac{m_i c}{e}\nabla \times [(\mathbf{u}_{i0} \bullet \nabla)\boldsymbol{\xi} - (\boldsymbol{\xi} \bullet \nabla)\mathbf{u}_{i0} - \boldsymbol{\xi} \times (\nabla \times \mathbf{u}_{i0})] \quad (4.6)$$

Note that (4.4) with (4.5) has the same form as the Frieman-Rotenberg equation [Eqs. (25) and (26) in Ref.[6]] except for difference in the relation between $\mathbf{B}_1$ and $\boldsymbol{\xi}$ shown here in (4.6). The first term of right side of (4.6) is the conventional one-fluid result; the second is from the Hall effect and the third is a combination of the Hall effect and ion flow.

To investigate the stability of the axisymmetric 2-D equilibrium, the ion displacement vector can be expressed as

$$\boldsymbol{\xi} = \hat{\boldsymbol{\xi}}(r, z)\exp[\, i(l\theta - \omega t)\,] \qquad (4.7)$$

where l is the azimuthal mode number and ω the angular frequency. Then (4.4) can be reduced to coupled equations for ξ_r and ξ_z where ω plays a role of the eigenvalue. If the equilibrium depends only on the radial coordinate, i.e. the axisymmetric, infinitely long 1-D equilibrium, the displacement vector can be expressed as

$$\boldsymbol{\xi} = \tilde{\boldsymbol{\xi}}(r)\exp[i(\mathbf{k}_W \bullet \mathbf{r} - \omega t)] \qquad (4.8)$$

where $\mathbf{k}_W = \hat{\boldsymbol{\theta}}\frac{l}{r} + \hat{\mathbf{z}}k$, k is the axial wave number, and $\hat{\mathbf{z}}$ is the unit vector in the axial direction. In this case, the coupled equations for ξ_r and ξ_z reduce to the following 4[th] order differential equation for the radial displacement ξ_r.

$$A_4 \frac{\partial^4 \xi_r}{\partial r^4} + A_3 \frac{\partial^3 \xi_r}{\partial r^3} + A_2 \frac{\partial^2 \xi_r}{\partial r^2} + A_1 \frac{\partial \xi_r}{\partial r} + A_0 \xi_r = 0 \qquad (4.9)$$

where the coefficients A_n ($n = 0,1,2,3,4$) are the functions of $\omega, l, k, \mathbf{B}_0(r), \mathbf{u}_{i0}(r)$. In several places in these coefficients appears the quantity W, which is defined as

$$W \equiv (\omega - \mathbf{k}_W \bullet \mathbf{u}_{i0})(\omega - \mathbf{k}_W \bullet \mathbf{u}_{e0}) - (\mathbf{k}_W \bullet \mathbf{B}_0)(\mathbf{k}_W \bullet \boldsymbol{\Omega}_{i0}) \qquad (4.10)$$

Here the difference between the one- and two-fluid models is apparent; in the one-fluid model, the above W function reduces to

$$W = (\omega - \mathbf{k}_W \bullet \mathbf{u}_{i0})^2 - (\mathbf{k}_W \bullet \mathbf{B}_0)^2 \qquad (4.11)$$

The first term in (4.11) represents the Doppler shift due to the ion flow. In a two fluid, however, Doppler shifts due to both the *ion* and *electron* flows appear. The effect of the electron flow can be significant in some configurations. The role of the second term in (4.11) is also critical in determining the stability because $\mathbf{k}_W \bullet \mathbf{B}_0 = 0$ defines magnetic rational surfaces. In a two fluid however, (4.10) not $\mathbf{k}_W \bullet \mathbf{B}_0$ but also the quantity $\mathbf{k}_W \bullet \boldsymbol{\Omega}_0$ appears. This implies that not only the magnetic rational surface but also the *vorticity* rational surface where $\mathbf{k}_W \bullet \boldsymbol{\Omega}_0 = 0$ affect the stability.

V. RESULTS

Although the coupled equations (3.7) and (3.8) can express various equilibria, we show here, as a preliminary result, a flowing ST-like high-beta equilibrium with infinite elongation, i.e. the equilibrium depends only on the radial coordinate r. In this configuration the axial external current flows in the center conductor and the plasma is confined inside the separatrix ($r_0 \le r \le a$). Figure 1 shows the result. The magnetic field and the flow are normalized by $B_{0z}(a)$ (the axial component of the magnetic field at r=a) and $V_A \equiv B_{0z}(a)\big/\sqrt{4\pi m_i n}$ (the reference Alfven speed based on that field), respectively. For this example, the size parameter $S_* \equiv a\omega_{pi}/c = 100$, and the total average beta $<\beta> \equiv <p_0>\big/<p_0 + B_0^2/8\pi> = 0.68$. Observe that (1) the magnetic structure is paramagnetic although the beta value of this equilibrium is high compared with conventional STs [10], and (2) both the azimuthal and axial flows are nearly the reference Alfven speed.

In order to compare the stability of flowing two-fluids, we are also studying the stability of static one-fluid equilibria. For the later we employ, instead of the conventional treatment [11], the new formalism which results from the incompressibility assumption as in (2.1). The validity of this stability formalism was tested for well-known configurations. Figure 2 shows the stability of static equilibria in the same geometry shown in Fig.1. The lowest order kink mode (azimuthal mode number $l = 1$) is investigated with various values of the axial wave number k . In the one-dimensional geometry l is analogous to the toroidal mode number and k is analogous to the poloidal mode number. In Fig.2-a, the horizontal axis shows the ratio $B_{0\theta}(r_0)/B_{0z}(a)$, representing the ratio of the azimuthal component of the magnetic field at $r = r_0$ produced by the axial external current to the axial component of the magnetic field at $r = a$. This is analogous to a ratio of toroidal-to-poloidal magnetic fields. The vertical axis shows the quantity $\left(B_{0\theta}(r) - B_V(r)\right)_{r=r_M}$ where $B_V(r)$ is the vacuum magnetic field produced by the current flowing in the center conductor and r_M is the magnetic axis location defined by $B_{0z}(r = r_M) = 0$. The total average beta $<\beta>$ is also shown in the right side. Red triangles indicate “unstable” and blue diamonds indicate “stable”. The orange, yellow and green triangles show “less and less unstable”. Hence, as the quantity $\left(B_{0\theta}(r) - B_V(r)\right)_{r=r_M}$, which is proportional to the center column current, increases, the system becomes more unstable. As the ratio $B_{0\theta}(r_0)/B_{0z}(a)$ increases, the system becomes more stable and is fully stabilized for sufficiently large values. Note that the ratio $B_{0\theta}(r_0)/B_{0z}(a) \approx 1$ for the equilibrium shown in Fig.1. The dependence of the growth rates on $B_{0\theta}(r_0)/B_{0z}(a)$ for $\left(B_{0\theta}(r) - B_V(r)\right)_{r=r_M} = 0.4$ (the total average beta ≈ 0.5) is shown in Fig.2-b. The growth rate is normalized by the quantity V_A/a . The yellow (green) curve shows the growth rates of the most (second most) unstable modes for a given equilibrium.

VI. SUMMARY

We have developed the spectral formalism of the stability analysis of flowing two-fluid plasmas. For axisymmetric equilibria, the coupled equations for the flux function and the ion stream function in the reduced case of uniform density are derived. This coupled system has four arbitrary surface functions while the Grad-Shafranov equation has only two. Minimum energy equilibria [7] with constant density is the subset of these equilibria. As a preliminary example of the two-fluid equilibria, a flowing ST-like high-beta equilibrium with infinite elongation was examined.

In the stability analysis, a new relation between the perturbed magnetic field and the ion displacement was derived. This relation is the natural generalization of Frieman and Rotenberg’s flowing one-fluid model. The coupled equations which describe the stability of the axisymmetric 2-D equilibria are derived. For 1-D equilibria these reduce to a 4th order differential equation for the radial displacement ξ_r . This equation includes the Doppler shifts due to both the ion and electron flows. These effects and the effects of the *vorticity* rational surface will have an important effect on stability. To study these effects is a focus of future work. To compare the stability of static and flowing equilibria, we will find equilibria with a broad range of flow speed and apply the present stability formalism to them.

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Fig1

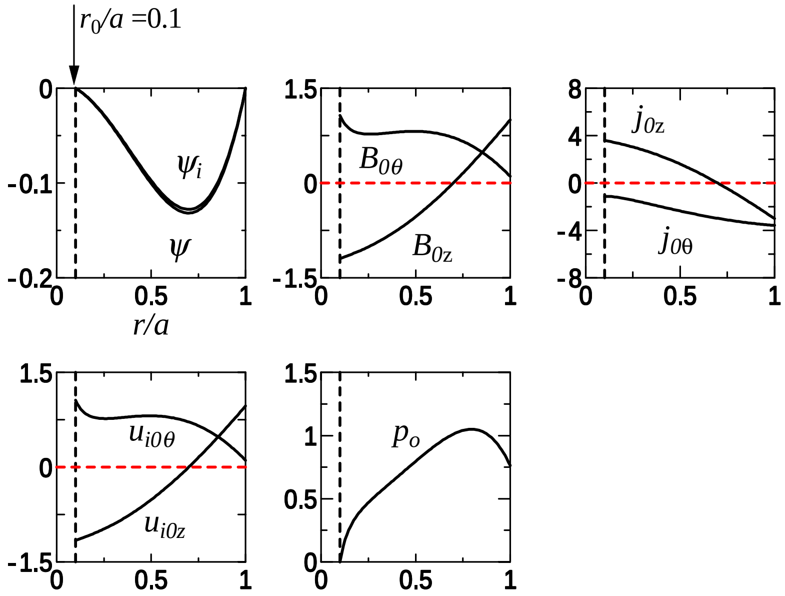

Fig.2-a

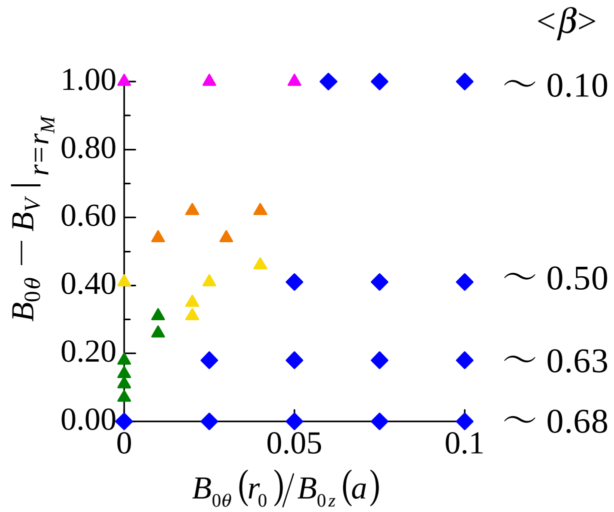

Fig.2-b