High-beta toroidal equilibrium with a strong shear flow – Double-Beltrami states —

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I. INTRODUCTION

A general solenoidal (divergence-free) vector field, such as a magnetic field or an incompressible flow, can be decomposed into an orthogonal sum of the Beltrami fields, $\nabla \times \boldsymbol{G}_i = \lambda_i \boldsymbol{G}_i$ [1]. Nonlinear dynamics of plasmas induce complex couplings among these Beltrami fields. In a single-fluid MHD plasma, however, the relaxed state is expressed by a single Beltrami magnetic field, resulting in the self-organization of a force-free equilibrium, which is the Taylor relaxed state [2].

In a two-fluid MHD plasma, by relating the velocity and the magnetic fields, a more general relaxed state is given by a double Beltrami field that is expressed by the superposition of two different Beltrami fields [3]. The double Beltrami fields, where velocity and magnetic fields are strongly coupled, include field structures far richer than the conventional single Beltrami fields. This new set of relaxed states, despite its simple mathematical structure, can express a variety of plasma states and explain interesting phenomena. Furthermore, it is shown that a generalized Bernoulli condition is satisfied simultaneously with the Beltrami condition. As the generalized Bernoulli conditions describe homogeneous distributions of the energy density, the Beltrami-Bernoulli states may follow from the concept of relaxed states. In the two-fluid MHD, the Beltrami-Bernoulli condition predicts the possibility of producing a very high-beta equilibrium, which is not allowed in the single Beltrami states. The double Beltrami field can be classified as a relaxed state that has a more complicated structure (higher energy) than the single Beltrami field.

II. BELTRAMI-BERNOULLI CONDITION

We start with reviewing the prototype equation for vortex dynamics. Let $\boldsymbol{\omega}$ be a three dimensional vector field representing a certain vorticity in $\boldsymbol{R}^3$. We consider an incompressible flow $\boldsymbol{U}$ that transports $\boldsymbol{\omega}$. When the circulation associated with the vorticity is conserved everywhere, this $\boldsymbol{\omega}$ obeys the equation

$$\frac{\partial}{\partial t}\boldsymbol{\omega} - \nabla \times (\boldsymbol{U} \times \boldsymbol{\omega}) = 0. \tag{1}$$

The general steady states of (1) are given by

$$\boldsymbol{U} \times \boldsymbol{\omega} = \nabla \varphi, \tag{2}$$

where φ is a certain scalar field, which physically corresponds to the energy density (pressure) in the original (decurled) equation.

The Beltrami condition, which demands alignment of vortices and flows, is expressed by

$$\boldsymbol{\omega} = \mu \boldsymbol{U}, \tag{3}$$

where μ is a certain scalar function. The Beltrami condition (3), thus, gives a special class of solution such that

$$\boldsymbol{U} \times \boldsymbol{\omega} = 0 = \nabla \varphi. \tag{4}$$

The former equality is the Beltrami condition, while the latter, demanding that the energy density is homogeneous, is a "generalized Bernoulli condition".

Normalizing the length by the ion skin-depth, the magnetic field $\boldsymbol{B}$ by an appropriate measure of the magnetic field and the fluid velocity $\boldsymbol{V}$ by the corresponding Alfvén speed, we can cast the electron ($j = 1$) and ion ($j = 2$) equations in a revealing symmetric vortex equation

$$\frac{\partial}{\partial t}\boldsymbol{\omega}_j - \nabla \times (\boldsymbol{U}_j \times \boldsymbol{\omega}_j) = 0 \quad (j = 1, 2) \tag{5}$$

in terms of a pair of generalized vorticities

$$\boldsymbol{\omega}_1 = \boldsymbol{B}, \quad \boldsymbol{\omega}_2 = \boldsymbol{B} + \nabla \times \boldsymbol{V},$$

the effective flows

$$\boldsymbol{U}_1 = \boldsymbol{V} - \nabla \times \boldsymbol{B}, \quad \boldsymbol{U}_2 = \boldsymbol{V},$$

and energy density of each fluid

$$\varphi_1 = \phi - p_e, \quad \varphi_2 = V^2/2 + \phi + p_i,$$

where p_e (or p_i) is pressure of electron (or ion) and ϕ is electro static potential. The simplest and perhaps the most fundamental equilibrium solution to (5) is given by the "Beltrami-Bernoulli condition". Assuming that a and b are constants, the Beltrami condition reads as a system of simultaneous linear equations in $\boldsymbol{B}$ and $\boldsymbol{V}$

$$\boldsymbol{B} = a(\boldsymbol{V} - \nabla \times \boldsymbol{B}), \tag{6}$$

$$\boldsymbol{B} + \nabla \times \boldsymbol{V} = b\boldsymbol{V}. \tag{7}$$

Combining (6) and (7) yields a second order partial differential equation

$$\nabla \times (\nabla \times \boldsymbol{B}) - (b - \tilde{a})\, \nabla \times \boldsymbol{B} + (1 - \tilde{a}b)\, \boldsymbol{B} = 0, \tag{8}$$

where $\tilde{a} = 1/a$. Denoting the curl derivative $\nabla\times$ by "curl", (8) is written as

$$(\mathrm{curl} - \lambda_+)(\mathrm{curl} - \lambda_-)\boldsymbol{B} = 0, \tag{9}$$

where

$$\lambda_\pm = \frac{1}{2}\left[(b - \tilde{a}) \pm \sqrt{(b + \tilde{a})^2 - 4}\right]. \tag{10}$$

Since the operators $(\mathrm{curl} - \lambda_\pm)$ commute, the general solution to the double curl Beltrami equation (9) is given by the linear combination of the two Beltrami fields. Let $\boldsymbol{G}_\pm$ be the Beltrami field such that

$$\begin{cases} (\mathrm{curl} - \lambda_\pm)\boldsymbol{G}_\pm = 0 & (\text{in } \Omega), \\ \boldsymbol{n} \cdot \boldsymbol{G}_\pm = 0 & (\text{on } \partial\Omega), \end{cases}$$

where Ω ($\subset \boldsymbol{R}^3$) is a bounded domain with a smooth boundary $\partial\Omega$ and $\boldsymbol{n}$ is the unit normal vector onto $\partial\Omega$. Then, for arbitrary constants $C_\pm$, the sum

$$\boldsymbol{B} = C_+\boldsymbol{G}_+ + C_-\boldsymbol{G}_-, \tag{11}$$

solves (9). The corresponding flow is given by

$$\boldsymbol{V} = (\lambda_+ + \tilde{a})\, C_+\boldsymbol{G}_+ + (\lambda_- + \tilde{a})\, C_-\boldsymbol{G}_-. \tag{12}$$

This equilibrium is called by double Beltrami field.

The Bernoulli condition $\varphi_j = \text{const}$ gives

$$\beta + V^2 = \text{const}, \tag{13}$$

where β is a conventional beta ratio that is given by $\beta = 2(p_e + p_i)$ in the normalized unit. This relation shows that the double Beltrami equilibrium is no longer zero-beta (force-free), but it can confine an appreciable pressure when an appreciable flow (in the Alfvén unit) is driven. To obtain such a fast flow in plasmas, a nonneutral condition is proposed, which can actually produce a self-electric field $\boldsymbol{E}$ in plasmas, causing strong $\boldsymbol{E} \times \boldsymbol{B}$ shear flow if we apply an appropriate magnetic field $\boldsymbol{B}$ there. For this purpose, a new method of toroidal non-neutral plasma trap has been developed using a proto-type device "Proto-RT" [4].

When a strong flow exists in addition to the current in a two-component plasma, the system must conserve two distinct helicities and the self-organized state becomes qualitatively different from the Taylor relaxed state [3,5]. The double Beltrami field may be classified as higher energy level than the single Beltrami field (see Fig. 1).

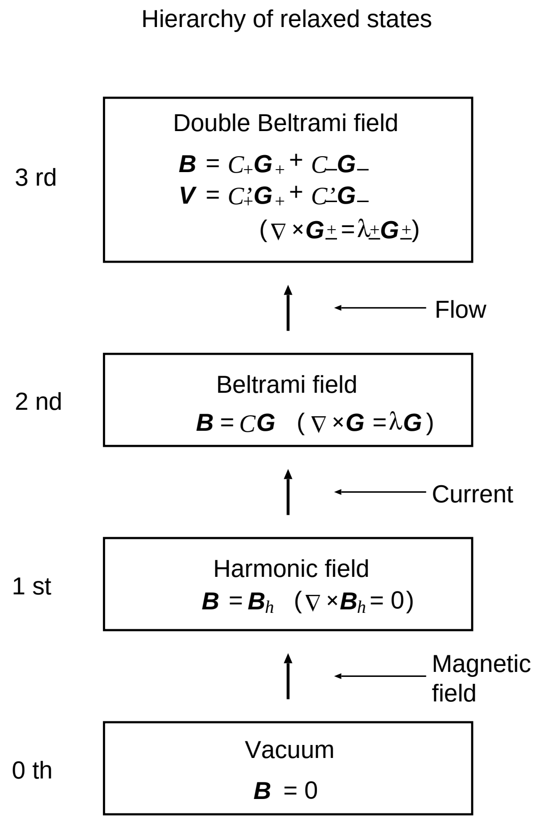

FIG. 1. Hierarchy of relaxed states. The absolute minimum energy state is the vacuum. Suppose that we apply a static magnetic field (harmonic field) and produce a plasma. Without applying any drive, the plasma will disappear and the system will relax into the harmonic magnetic field. If we drive the plasma current to sustain the total helicity, the plasma relaxes into the Taylor state and achieves the Beltrami field. We may also drive a flow (or inject charge). Then the relaxed state will be the double Beltrami field.

III. HIGH-BETA TOROIDAL EQUILIBRIUM

Due to the simple mathematical structure of the double Beltrami fields, it is rather easy to find analytical solutions of various equilibria in slab or cylindrical geometry. By choosing an appropriate set of parameters, we can construct very high beta solutions with producing a large flow velocity $\boldsymbol{V}$.

In this section, we present a coupled Grad-Shafranov equation of the double Beltrami equilibrium. We consider axisymmetric two-dimensional equilibria. Following the basic idea of formulating the Grad-Shafranov equation, we use the Clebsch representations of divergence-free axisymmetric ($\partial_\theta = 0$) vector function in cylindrical (r, θ, z) coordinates. We write $\boldsymbol{B}$ and $\boldsymbol{V}$ in a contravariant-covariant combination form

$$\boldsymbol{B} = \nabla\Psi(r,z) \times \nabla\theta + rB_\theta(r,z)\,\nabla\theta, \tag{14}$$

$$\boldsymbol{V} = \nabla\Phi(r,z) \times \nabla\theta + rV_\theta(r,z)\,\nabla\theta, \tag{15}$$

where Ψ (or Φ) is the flux function (or the stream function) of r and z, and B_θ (or V_θ) is the azimuthal magnetic (or velocity) field depending on r and z.

Using these expressions in the double Beltrami condition (6) and (7), and comparing each component, we get a coupled Grad-Shafranov equation;

$$-\mathcal{L}\begin{pmatrix}\Psi\\ \Phi\end{pmatrix} = \begin{pmatrix}1-\tilde{a}^2 & \tilde{a}-b\\ b-\tilde{a} & 1-b^2\end{pmatrix}\begin{pmatrix}\Psi\\ \Phi\end{pmatrix} - \begin{pmatrix}\tilde{a}C_1 + C_2\\ C_1 + bC_2\end{pmatrix}, \tag{16}$$

where C_1 and C_2 are constants, and $\mathcal{L}$ is the familiar Grad-Shafranov operator,

$$-\mathcal{L} = r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2}. \tag{17}$$

Solutions of high-beta toroidal equilibrium are obtained by iterating numerical analysis of (16). We obtain a solution by finding functions $\Psi_0^{(n+1)}$ and $\Phi_0^{(n+1)}$ that satisfy the following equation with $\Psi_0^{(n+1)} = \Psi_0^{(n)}$ and $\Phi_0^{(n+1)} = \Phi_0^{(n)}$:

$$-\mathcal{L}\begin{pmatrix}\Psi^{(n+1)}\\ \Phi^{(n+1)}\end{pmatrix} = \begin{pmatrix}1-\tilde{a}^2 & \tilde{a}-b\\ b-\tilde{a} & 1-b^2\end{pmatrix}\begin{pmatrix}\Psi^{(n)}\\ \Phi^{(n)}\end{pmatrix} - \begin{pmatrix}\tilde{a}C_1 + C_2\\ C_1 + bC_2\end{pmatrix} + \begin{pmatrix}\Psi_v^{(n+1)}\\ \Phi_v^{(n+1)}\end{pmatrix}. \tag{18}$$

We note that the "vacuum fields" satisfying

$$-\mathcal{L}\Psi_v = 0, \tag{19}$$

$$-\mathcal{L}\Phi_v = 0, \tag{20}$$

can be included both in Ψ and Φ as inhomogeneous terms. Choosing appropriate vacuum fields, we can control the radial position and the shape of the toroidal equilibrium.

We set the following boundary conditions to obtain free-boundary equilibria. A conductive vessel is not considered, so that Ψ and Φ go to constants at infinity. These constants are chosen so that $\Psi = 0$ and $\Phi = 0$ at the plasma boundary, which are defined by the contour of Ψ and Φ that touch the limiter, viz,

$$\max_{(r,z)\in \mathrm{limiter}} \Psi(r,z) = 0, \tag{21}$$

$$\max_{(r,z)\in \mathrm{limiter}} \Phi(r,z) = 0. \tag{22}$$

First we compare a numerical solution to (16) in large aspect ratio with an analytical solution in cylindrical geometry for the confirmation of the correctness of our code. When the toroidal plasma current and the total toroidal plasma vorticity are geiven as I_t and Ω_t, we can obtain an analytical solution in cylindrical geometry as follows,

$$\hat{\Psi} = R\left[\frac{C_+}{\lambda_+}\{1 - J_0(\lambda_+ r)\} + \frac{C_-}{\lambda_-}\{1 - J_0(\lambda_- r)\}\right], \tag{23}$$

$$\hat{\Phi} = R\left[(\tilde{a}+\lambda_+)\frac{C_+}{\lambda_+}\{1 - J_0(\lambda_+ r)\} + (\tilde{a}+\lambda_-)\frac{C_-}{\lambda_-}\{1 - J_0(\lambda_- r)\}\right], \tag{24}$$

and

$$C_+ = \frac{\Omega_t - (\tilde{a}+\lambda_-)\,I_t}{2\pi r_0\,(\lambda_+ - \lambda_-)\,J_1(\lambda_+ r_0)}, \tag{25}$$

$$C_- = \frac{-\Omega_t + (\tilde{a}+\lambda_+)\,I_t}{2\pi r_0\,(\lambda_+ - \lambda_-)\,J_1(\lambda_- r_0)}, \tag{26}$$

where where J_0 and J_1 are the ordinary Bessel functions, and R and r_0 correspond to major- and minor rudius.

We compare Ψ and Φ given by (16) with $\hat{\Psi}$ and $\hat{\Phi}$ in Fig. 2, where we set the aspect ratio is 100.

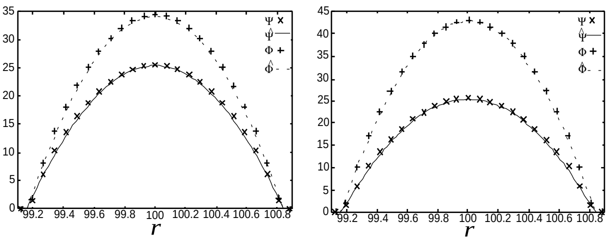

FIG. 2. Comparison between numerical and analytical solution.

In this figure, the numerical solution Ψ (or Φ) is expressed by a point $\times$ (or $+$), and analytical solution $\tilde{\Psi}$ (or $\tilde{\Phi}$) is drawn by a solid (or dashed) line. We can ascertain that the numerical and the analytical solutions coincide well.

In this equilibrium, since fast flow is induced, we need to care about shock formation. Therefore, using an MHD shock condition called by evolutionary condition, we will discuss a shock formation for the double Beltrami equilibrium in toroidal system. The evolutionary condition indicates that, if the plasma flow velocity $V = |\boldsymbol{V}|$ is larger than the slow wave (V_s) and smaller than the Alfén wave (V_{Ax}), the slow shock may be created, and if $V_f < V$, the fast shock may appear [6]. Here the phase velocities of each wave are defined by

$$V_{Ax} = V_A \cos\theta = \sqrt{\frac{B_x^2}{\rho\mu_0}}, \tag{27}$$

$$V_f = \Big\{(1/2)\left[V_A^2 + C_s^2 + \sqrt{\left(V_A^2 + C_s^2\right)^2 - 4V_{Ax}^2 C_s^2}\right]\Big\}^{1/2}, \tag{28}$$

$$V_s = \Big\{(1/2)\left[V_A^2 + C_s^2 - \sqrt{\left(V_A^2 + C_s^2\right)^2 - 4V_{Ax}^2 C_s^2}\right]\Big\}^{1/2}, \tag{29}$$

where V_A is the Alfvén velocity, C_s is the sound velocity and θ is the angle between the background flow (i.e., x-direction) and magnetic field. The V_{Ax} is the Alfvén velocity parallel component to the plasma flow.

Finally, we discuss an equilibrium with plasma flow in small aspect ratio toroidal geometry. Setting the aspect ratio is 3, we solve the basic equations (16). The results are shown in Fig. 3 - Fig. 4. We plot the contour of Ψ and Φ in Fig. 3, β profile in Fig. 4 (a) and the plasma flow $|\boldsymbol{V}|$ and local V_s, V_{Ax} and V_f at each point in Fig. 4 (b).

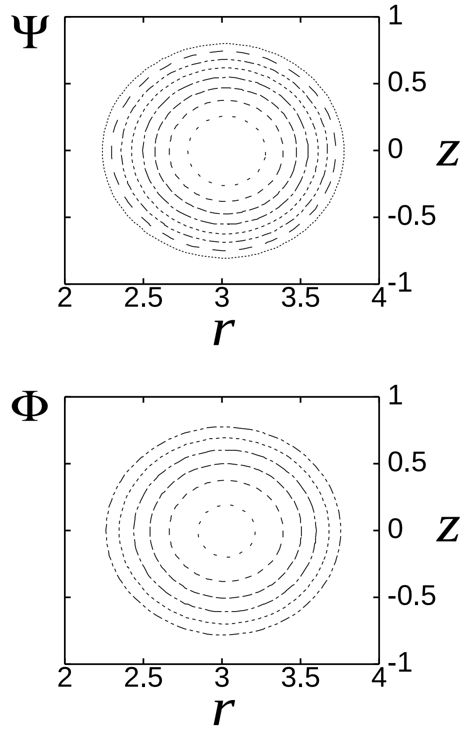

FIG. 3. The contours of Ψ and Φ in toroidal equilibrium of high-beta double Beltrami field.

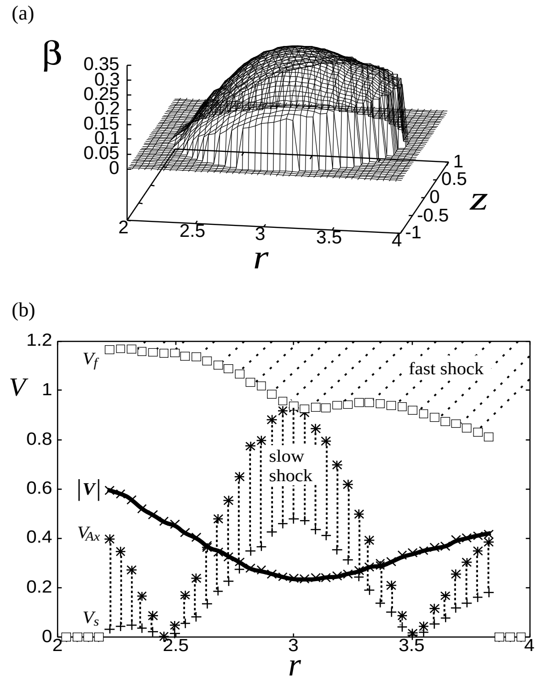

FIG. 4. (a) The beta profile, (b) Plasma flow and the regions of shock formation in toroidal equilibrium of high-beta double Beltrami field.

Now, we have two problems to be solved.

(i) There is a jump of beta (pressure) at the boundary of plasma.

(ii) There is a possibility that a slow shock is created in the plasma.

We may solve (ii) easily by setting the Beltrami parameters $\tilde{a}$ and b as proper values. Here, we showed the result only in the case of $\tilde{a} = b$. Using the analytical solutions (23) and (24), the result in the case of $\tilde{a} \neq b$ is plotted in Fig. 5 and Fig. 6, where we set $\tilde{a} = -1$, $b = 0.5$ and $\tilde{a} = -1$, $b = 1$. In both figures, the beta profile are plotted in (a), the flow profile in (b). From these results, we can say that a high-beta equilibrium can exist avoiding a shock or minimizing this effect by setting proper Beltrami parameters.

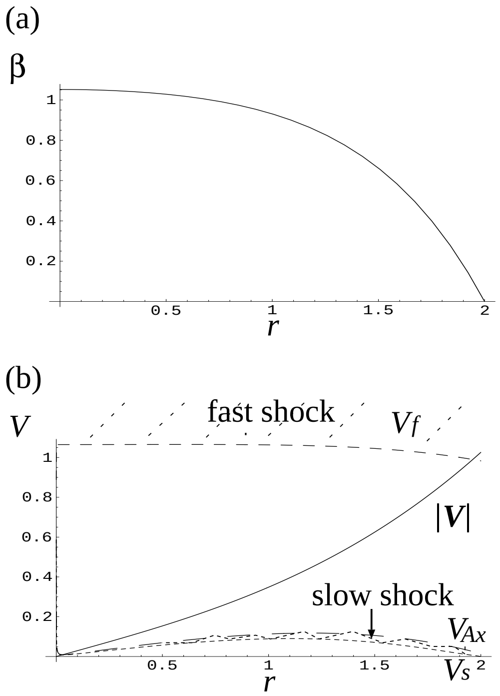

FIG. 5. Plasma flow and the regions of shock formation in cylindrical equilibrium of high-beta double Beltrami field when $\tilde{a} = -1$, $b = 0.5$.

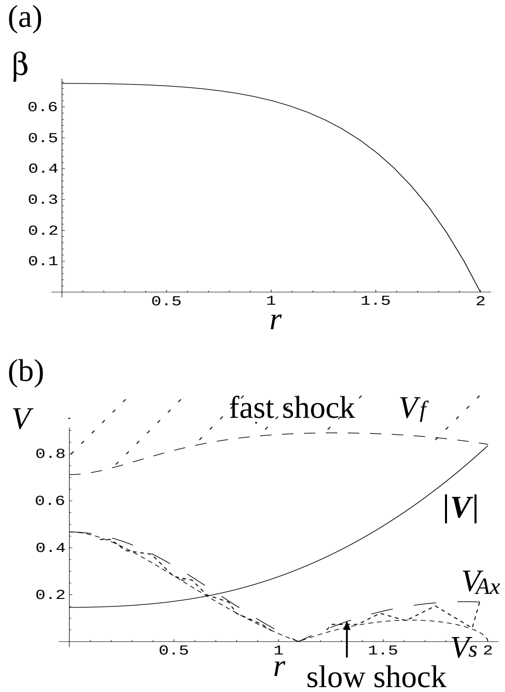

FIG. 6. Plasma flow and the regions of shock formation in cylindrical equilibrium of high-beta double Beltrami field when $\tilde{a} = -1$, $b = 1$.

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