Analytic Flowing Equilibria of Compact Toroids

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Abstract

Two-fluid flowing equilibria are explored, focusing on "stationary-energy" states with uniform density. Attention is limited to on compact toroids and to the region inside the separatrix. The equilibria are fall into two classes, coupled and decoupled, referring to the linkage between the magnetic field and the flows. The *coupled* class is force-free and may or may not have large flows. Spheromaks are in this class. The *decoupled* class has Alfvenic poloidal flows and generally have high β. FRCs are in this class. Both classes occupy particular "allowed" domains in "helicity space," a 2D map with the electron and ion helicities as coordinates. The allowed domains for the two classes overlap. Throughout the domain allowed to decoupled equilibria, they are energetically preferred. Coupled equilibria are only expected in the domain forbidden to the decoupled class. This bifurcation may explain the FRC-spheromak bifurcation observed in experiments. Analytic equilibria are found that apply to both classes.

I. Introduction to the two-fluid model

The possibility of natural plasma states with finite pressure has re-emerged in studies of the two-fluid model,[1,2,3] which is more general than the MHD formulation. A key feature of these states is significant flow. As such they constitute a new class of plasma, contrasting with standard MHD models of fusion plasmas in which the flow energy is negligible compared with the magnetic energy. We present the first detailed analysis of 2D stationary-energy states of a flowing two-fluid. To facilitate the analysis, three simplifications are made: (1) stationary-energy state (which is a sufficient condition for stability); (2) uniform density; and (3) angular momentum conservation is ignored.

A two-fluid has a natural length scale,[2,3] the ion skin depth $\mathrm{l}_i = (m_i c^2/4\pi e^2 n)^{1/2}$ in contrast to MHD which has no natural scale. Flows and fields for each fluid species are connected by the canonical momentum $\mathbf{P}_\alpha = m_\alpha \mathbf{u}_\alpha + q_\alpha \mathbf{A}/c$ ($\alpha = i,e$). A two-fluid has ideal magnetofluid invariant, the generalized helicities, $K_\alpha = (c^2/8\pi q_\alpha^2)\int d\tau\ \mathbf{P}_\alpha \cdot \nabla\times\mathbf{P}_\alpha$ one for each species ($\alpha = i,e$). The global angular momentum, $L = \int d\tau\, m_i n r u_{i\theta}$ is also invariant in axisymmetric systems with suitable boundary conditions. Relaxation theory postulates that weakly dissipative systems approach the state of maximum *entropy* of the ideal (nondissipative) system subject to its constraints. These are equivalent to the minimum energy states subject to the same constraints. The energy minimized is the *organized* energy, called the *magnetofluid* energy, $W_{MF} = \int d\tau \left(m_i n u_i^2/2 + B^2/8\pi\right)$. The Euler equations for the stationary energy state[2] are $\mathbf{u}_\alpha = \Omega r\hat{\boldsymbol{\theta}} + \lambda_\alpha (c^2/4\pi e^2 n)\nabla\times\mathbf{P}_\alpha$ where the parameters λ_e, λ_i, and Ω are the Lagrange multipliers associated with the two helicity and the angular momentum constraints, respectively. Familiar reduced cases are limits of the two-fluid model. A *simple fluid* has a single helicity invariant, the *fluid* helicity, found by setting the $e = 0$ in K_i; its stationary-energy states are Beltrami states. An *MHD fluid* has a single helicity invariant, the magnetic helicity found by setting $m_e = 0$ in K_e; its stationary states are Taylor states. Double-Beltrami states[3] are stationary states of a two fluid where angular momentum conservation is ignored.

II. Analysis of two-fluid equilibria

Assume a stationary-energy state with constant density, and ignore angular momentum conservation. Then the equilibrium equations are[4]

$$\overleftrightarrow{\mathbf{D}}(\Delta^*)\begin{Bmatrix} c\psi/4\pi e \\ \psi_i/\lambda_i \end{Bmatrix} = 0 \qquad \overleftrightarrow{\mathbf{D}}\left(\Delta^*\right) = \begin{bmatrix} \Delta^* + \lambda_e^2 - 1/\ell_i^2 & \lambda_i\lambda_e + 1/\ell_i^2 \\ -(\lambda_i\lambda_e + 1/\ell_i^2) & \lambda_i^2(\ell_i^2\Delta^* - 1) + 1/\ell_i^2 \end{bmatrix} \quad (1,2)$$

where Δ^* is the Grad-Shafranov operator, and ψ,ψ_i are the stream functions for the poloidal field and flow. Auxiliary equations govern the toroidal components. For uniform density the pressure is given by Bernoulli's equation $p + m_i n u_i^2/2 = const$. The two parameters λ_e,λ_i are Lagrange multipliers associated with the helicities constraints. Both stream functions obey a modified Helmholtz equation, $(\Delta^* + \Lambda^2)\{\psi,\psi_i\} = 0$ with eigenvalue Λ. Note that the boundary geometry requires that the eigenvalue be one member of a discrete set (infinite) of real numbers. Nontrivial solutions exist only if the determinant of the coefficient matrix vanishes

$$\| \overleftrightarrow{\mathbf{D}}(-\Lambda^2) \| = 0 \quad (3)$$

Two classes of equilibria appear: if the off-axis elements of $\overleftrightarrow{\mathbf{D}}(-\Lambda^2)$ are nonzero the flows and fields are *coupled*; if the off-axis elements are zero the fields and flows are *decoupled*. The fields and flows for these two classes are as follows

$$\mathbf{B} = \alpha\Lambda\psi\frac{\hat{\boldsymbol{\theta}}}{r} + \frac{\hat{\boldsymbol{\theta}}}{r}\times\nabla\psi\,; \qquad \mathbf{u}_i = -\frac{c}{4\pi en}\left[(\alpha\lambda_e + \Lambda)\Lambda\psi\frac{\hat{\boldsymbol{\theta}}}{r} + (\lambda_e + \alpha\Lambda)\frac{\hat{\boldsymbol{\theta}}}{r}\times\nabla\psi\right] \quad (4,5)$$

The parameters $\lambda_e,\lambda_i,\alpha$ are connected by the characteristic equation:

$$\text{Coupled:} \quad \lambda_i = -\frac{\lambda_e + \Lambda}{1 + \Lambda(\lambda_e + \Lambda)\ell_i^2}\,; \qquad \alpha = 1; \qquad \lambda_e = \text{free} \quad (6a)$$

$$\text{Decoupled:} \quad \lambda_e = (1/\ell_i)[1 + \Lambda^2\ell_i^2]^{1/2}\,; \quad \lambda_i = -(1/\ell_i)[1 + \Lambda^2\ell_i^2]^{-1/2}\,; \quad \alpha = \text{free} \quad (6b)$$

The properties of the two classes of equilibria are as follows. (1) *Coupled class*. (a) The current density is aligned with the magnetic field, i.e. these states are *force-free*. (b) Pressure nonuniformities arise only from inertial effects. (c) The flow is also aligned with the magnetic field. (d) The flow velocity may range from zero to Alfvenic. (2) *Decoupled class*. (a) The current is *not* aligned with the magnetic field so that the field is *diamagnetic*, and high β is allowed. (b) Only the poloidal parts of the flow and field are aligned. (c) The poloidal flow speed is *Alfvenic*.

III. Global integrals of flowing equilibria

Given the foregoing solutions the global integrals can be found. The objective is to eliminate the free parameter (λ_e or α) in favor of the helicities in order to find the express the organized energy in the functional form $W_{MF} = W_{MF}(K_e,K_i,\Lambda)$. The two helicities can be thought of as the coordinates in a two-dimensional map of helicity space; on this map the domains of the equilibrium types and their properties can be classified. The function $W_{MF}(\ldots)$ is different for the two classes of equilibria. In the coupled class

$$\textit{coupled}: \qquad W_{MF} = \left[|\Lambda| l_i + \frac{1}{|\Lambda| l_i}\left(\sqrt{K_i/K_e} - 1\right)^2\right]\frac{|K_e|}{l_i} \tag{7}$$

Note that K_i and K_e must have the same sign; the case of opposite signs is forbidden to coupled equilibria. Only the positive square root is shown here; the negative square root case is also a solution but it has much higher energy; the case shown is *energetically favorable*. The first and second terms in the square brackets represent the magnetic and flow energies, respectively. At a critical value $|\Lambda| = \Lambda_C \equiv (\sqrt{K_i/K_e} - 1)/l_i$ the two energies are equal, i.e. energy equipartition. For $|\Lambda| > \Lambda_C$ the magnetic energy is larger and for $|\Lambda| < \Lambda_C$ the flow energy is larger. The flow velocity is

$$\textit{coupled}: \qquad \mathbf{u}_i = -V_A \frac{\sqrt{K_i/K_e} - 1}{\Lambda l_i}\frac{\mathbf{B}}{B} \tag{8}$$

where $V_A = B/\sqrt{4\pi m_i n}$ is the local Alfven speed. In the *decoupled class*

$$W_{MF} = \frac{1}{l_i}\left|\frac{K_i}{\sqrt{1+\Lambda^2 l_i^2}} - K_e\sqrt{1+\Lambda^2 l_i^2}\right| \tag{9}$$

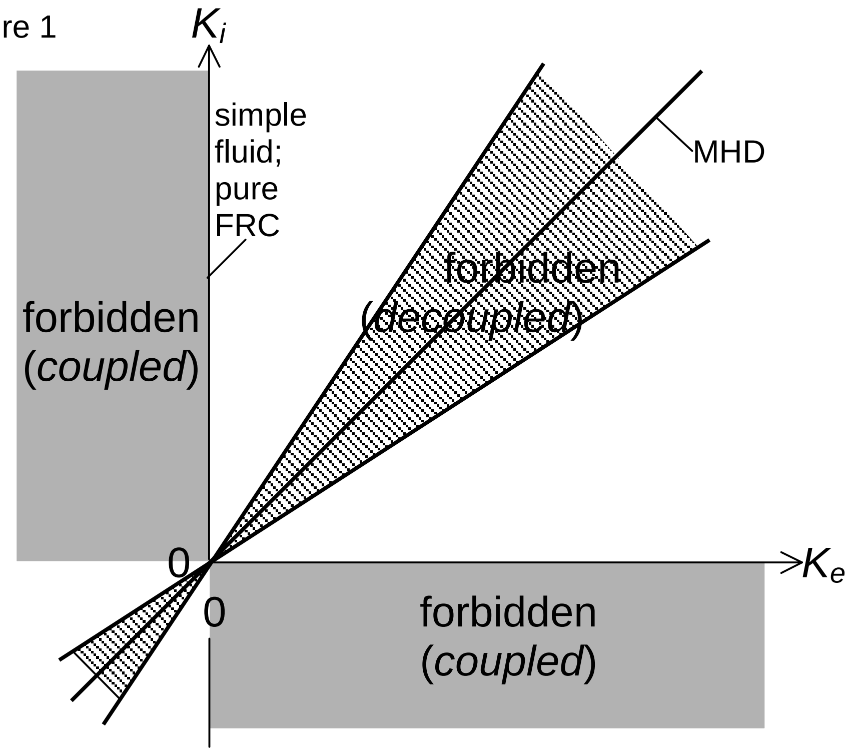
There is an approximate equipartition between flow and field energy throughout most of the allowed region. In the decoupled class *only* the poloidal parts of the flow and field vectors are aligned:

$$\mathbf{u}_{ip} = -\left[\sqrt{1+\Lambda^2 l_i^2} + O(\Lambda^2 l_i^2)\right]V_{Ap}\frac{\mathbf{B}_p}{B_p} \tag{10}$$

Here "p" = poloidal, and $V_{Ap} = B_p/(4\pi m_i n)^{1/2}$ is the "poloidal" Alfven speed.

From the analysis of the global integrals, note the following properties. (1) *Coupled equilibria*. (a) Both helicities must have the same sign. Thus the allowed combinations of K_e-K_i are the I and III quadrants of K_i vs K_e space. (b) The flow is *parallel* (*antiparallel*) to the field if the helicities are positive (negative). The flow is quite large (compared to the Alfven speed) unless the two helicities are nearly equal. (c) Equipartition between magnetic and flow energies occurs at a critical value Λ_C that depends on K_i/K_e. (2) *Decoupled equilibria*. (a) The current density and magnetic field vectors are not aligned; thus these equilibria are *not* force free. FRCs are an example of decoupled equilibria. (b) The forbidden region in K_i vs K_e space is a narrow "wedge" around the $K_i = K_e$ line. The helicities need not have the same sign. Moreover, the electron helicity may vanish without forcing the uninteresting case of no magnetic fields; indeed, FRCs have $K_e = 0$. (c) Only the poloidal parts of the ion flow and magnetic field are aligned. (d) An approximate equipartition between magnetic and flow energies occurs in most of K_i vs K_e space.

Figure 1



Equilibrium types can be portrayed on a helicity map, a two-dimensional space with the two helicities as coordinates as shown in Fig. 1. Mainly only the first and fourth quadrants are shown since the second and third ($K_e < 0$) are identical, although inverted. The domains are bounded by rays from the origin, i.e. lines of constant K_i/K_e. *Coupled* equilibria are forbidden in the shaded domain. Two familiar reduced cases lie in the allowed: MHD states (Taylor states) are on the $K_i = K_e$ line, and simple fluids (Beltrami states) are on the K_i axis. *Decoupled* equilibria are forbidden only in the narrow wedge centered on the $K_i = K_e$ line (hatched region in Fig. 1). Clearly the allowed domain for the decoupled class is much broader than that for the coupled class. FRCs lie on the K_i axis, the same line occupied by Beltrami states; of course, FRCs are in the decoupled class while Beltrami states are in the coupled class.

The allowed domains for coupled and decoupled classes overlap on the helicity map (regions neither shaded or hatched in Fig. 1). In the overlap regions *either* class is allowed. Relaxation processes are likely to select the class with lower energy. Comparing Eqs. (7,9) it is easily shown that throughout its allowed region, the decoupled class has lower W_{MF}. The coupled equilibrium then are expected only in the region marked "forbidden decoupled" in Fig. 1. This represents a *bifurcation* between the two classes of equilibria. Since one class has high-β and the other is force-free, this may explain the FRC-spheromak bifurcation observed in the TS-3 experiments.[5]

IV. Analytic equilibrium structure

Given the boundary geometry, the eigenvalue Λ must be one of a set of discrete values, each representing a separate equilibrium. It is useful, however, to treat Λ as a free parameter and allow the separatrix boundary to adjust accordingly. This allows simple analytic solutions for the stream function structure $\psi(r,z)$, $\psi_i(r,z)$. A simple solution is

$$\psi \quad = \quad B_0 r \frac{\Lambda J_1(\Lambda_\perp r)\cos(\Lambda_\| z) - \Lambda_\perp J_1(\Lambda r)\cos(\Lambda_\| b)}{\Lambda\Lambda_\perp[1-\cos(\Lambda_\| b)]} \tag{12}$$

where $B_0 \equiv |B|(r=0,z=0)$ where $\Lambda_\perp$, $\Lambda_\|$ are determined by

$$\Lambda_\perp^2 + \Lambda_\|^2 \;=\; \Lambda^2; \qquad \Lambda J_1(\Lambda_\perp a) - \Lambda_\perp J_1(\Lambda a)\cos(\Lambda_\| b) \quad = \quad 0 \tag{13}$$

A closed separatrix requires $\pi/2 \le \Lambda_\| b \le \pi$. The parameter $\Lambda_\perp \approx 3.8/a$ where 3.8 is near first zero of the J_1 Bessel function. Superficially the equilibrium structures resemble static MHD equilibria but there are important differences that will be explored elsewhere.

References

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